SVD Pseudo-inverse and Application to Image Reconstruction from Projections

(SVD Pseudo-inverse를 利用한 映像 再構成)

沈 英 錫* 金 昇 弼**
(Shim, Young Serk, Kim, Seung Pil)

要 約

Singular value decomposition을 通한 pseudo-inverse를 斷層映像 再構成에 利用하였다. 本 論文에서는 SVD pseudo-inverse를 利用한 truncated inverse filter와 Scalar Wiener filter에 對하여 檢討하고 各各에 對하여 統計的 側面에서의 最適化가 연구되었다. 이러한 方法은 信號와 雜音間에 trade-off를 期함으로써 再構成 問題에 恒常 뒤따르는 ill-conditioning 現象을 克服할 수 있다. 本 論文을 通하여 構成된 filter의 成能을 確認하기 위하여 컴퓨터를 利用한 simulation이 이루어졌으며 그 結果 再構成된 映像은 滿足할 만 하였다.

Abstract

A singular value decomposition (SVD) pseudo-inversion method has been applied to the image reconstruction from projections. This approach is relatively unknown and differs from conventionally used reconstruction methods such as the Forier convolution and iterative techniques. In this paper, two SVD pseudo-inversion methods have been discussed for the search of optimum reconstruction and restoration, one using truncated inverse filtering, the other scalar Wiener filtering. These methods partly overcome the ill-conditioned nature of restoration problems by trading off between noise and signal quality. To test the SVD pseudo-inversion method, simulations were performed from projection data obtained from a phantom using truncated inverse filtering. The results are presented together with some limitations particular to the applications of the method to the general class of 3-D image reconstruction and restoration.

I. Introduction

Recently the SVD method has gained popularity in a wide range of signal processing areas. It has become a conventional and well-known method in image restoration problems.[1-3]

In this paper a SVD pseudo-inversion method is applied to image reconstruction from projections which is believed to be relatively new.[4-6] The termination criteria and scalar Wiener filter function which minimize the residual mean square error under the given structural constraints are derived, and SVD method developed was then applied to the image reconstruction problems in computerized tomographic system through computer simulation under a coarse modeling of projection operation.

*正會員, **準會員, 韓國科學院 電氣 및 電子工學科
(Dept. of Electrical Science, KAIS)
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II. Image model and Restoration problems

For the stochastic consideration of restoration problems, image is usually modelled as a combination of mean signal and zero mean stationary random signal. In this paper, zero mean signal is denoted by symbol $\underline{f}$ and it is considered as an original image to be restored.

Using a discretized model, one can represent a linear degradation superimposed by additive noise as,

$$\underline{g} = [W]\,\underline{f} + \underline{n}\,, \tag{1}$$

where $\underline{f}$ and $\underline{g}$ are column stacked vectors of original and the degraded images, respectively. [W] is a system matrix which describes the degradation parameters of a system. And $\underline{n}$ is a column vector of additive noise which comes from sensors. Here $\underline{f}$ is an N-dimensional vector, $\underline{g}$ and $\underline{n}$ are M-dimensional.

With a given observation $\underline{g}$, the goal is to design a restoration system which produces a minimum length least square (MLLS) solution of the restoration problem. The first intuitively attractive approach is to use a Penrose-Moore pseudo-inverse $[W]^+$. Then restored image $\hat{\underline{f}}$ may be written as

$$\hat{\underline{f}} = [W]^+\underline{g}\ . \tag{2}$$

Eq. (2) gives an MLLS solution of the restoration problem when the noise term is zero.

But in the presence of noise, $\hat{\underline{f}}$ becomes

$$\hat{\underline{f}} = [W]^+[W]\underline{f} + [W]^+\underline{n}\ , \tag{3}$$

where $[W]^+\,\underline{n}$ may be considered as a perturbation to an MLLS solution. Unfortunately in most restoration problems, there exists an ill-conditioned or a singular nature. The perturbation term of restored image due to noise results in an obscured and corrupted image. To overcome this perturbation problem, a SVD method in conjunction with the pseudo-inversion will be considered to adopt a stochastic pseudo-inverse $[W]_s^+$ and a truncated pseudo-inverse $[W]_t^+$.

III. SVD of System Matrix [W] and Restoration Filter

Any M by N matrix [W] can be decomposed as[6],

$$[W] = [U]\ [S_G]\ [V]^t\,, \tag{4}$$

where [U] and [V] are M by M and N by N unitary matrices, $[S_G]$ is an M by N matrix with only ordered nonzero G-diagonal elements, $[\lambda_i^{1/2},\ i = 1, 2, \ldots, G]$, which represent the singular values of the system [W], respectively. And G is a rank of [W]. Here the superscript t is used to denote a matrix transpose.

1. Truncated SVD pseudo-inverse filter

Using matrix outer product expansion, Eq. (4) becomes

$$[W] = \sum_{i=1}^{G} \lambda_i^{1/2}\,\underline{u}_i\,\underline{v}_i^t\ , \tag{5}$$

where $\underline{u}_i$ and $\underline{v}_i$ are the i-th column vectors of [U] and [V], respectively, and $\lambda_i^{1/2}$ is the i-th singular value of [W]. Furthermore the pseudo-inverse of [W] is given by

$$[W]^+ = \sum_{i=1}^{G} \lambda_i^{1/2}\,\underline{v}_i\,\underline{u}_i^t\ . \tag{6}$$

As previously mentioned in section II, use of Eq. (6) in image restoration still possess a noise amplification because of the ill-conditioned nature of the restoration problem. The ill-conditioning originates from relatively smaller singular values to the other ones. So $[S_G]$ is modified by setting the smaller singular values to zeros, that is to choose the number of terms in Eq. (6) less than G. Determination of optimum termination index ℓ is subject to further discussions. Following the above approach one can have a truncated pseudo-inverse $[W]_t^+$ as,

$$[W]_t^+ = \sum_{i=1}^{\ell} \lambda_i^{-1/2}\,\underline{v}_i\,\underline{u}_i^t\ . \tag{7}$$

Using Eq. (7) a restored image $\hat{\underline{f}}$ can be written as,

$$\begin{aligned}\hat{\underline{f}} &= [W]_t^+\underline{g}\\ &= \sum_{i=1}^{\ell}\lambda_i^{-1/2}\,\underline{v}_i\,\underline{u}_i\,[\sum_{i=1}^{G}\lambda_j^{1/2}\,\underline{u}_j\,\underline{v}_j^t\,\underline{f} + \underline{n}]\\ &= \sum_{i=1}^{\ell}\sum_{j=1}^{G}\lambda_i^{-1/2}\,\underline{v}_i\,(\underline{u}_i^t\,\underline{u}_j)\,(\underline{v}_j^t\,\underline{f}) + \sum_{i=1}^{\ell}\lambda_i^{-1/2}\,\underline{v}_i\,\underline{u}_i^t\,\underline{n})\\ &= \sum_{i=1}^{\ell}[f_i\,\underline{v}_i + \lambda_i^{-1/2}\,n_i\,\underline{v}_i\,]\end{aligned} \tag{8}$$

where $f_i = \underline{f}^t\,\underline{v}_i$, $n_i = \underline{n}^t\,\underline{u}_i$, and ℓ denotes termination index previously mentioned. Then residual error vector $\underline{r}$, which is a difference between original and reconstruction, becomes

$$\underline{r} = \underline{f} - \hat{\underline{f}} = -\sum_{i=1}^{\ell}\lambda_i^{-1/2}\,n_i\,\underline{v}_i + \sum_{i=\ell+1}^{N} f_i\,\underline{v}_i \tag{9}$$

From Eq(9) it can be seen that by introducing modified SVD, noise effects can be reduced with some loss of signals which are the components of υ, where υ is a span of $\underline{v}_{\ell+1}, \underline{v}_{\ell+2}, \ldots, \underline{v}_G$. If ℓ is increased, the first summation in Eq.(3) will be closer to the original object, but the noise effects become so large that proper restoration will be prohibited. Therefore one has to make a reasonable compromise between these two effects, namely noise and resolution.

Proper selection of termination index ℓ can be made using the following procedure. From Eq.(9) the norm square of the residual vector is

$$\|\underline{r}\|^2 = \sum_{i=1}^{\ell} \lambda_i^{-1} n_i^2 + \sum_{i=\ell+1}^{N} f_i^2 \tag{10}$$

And $\underline{n}$ and $\underline{f}$ can be decomposed as,

$$\underline{n} = \sum_{i=1}^{M} d_i \phi_i^M , \quad \underline{f} = \sum_{i=1}^{N} a_i \phi_i^N , \tag{11}$$

where ϕ_i^M'S and ϕ_i^N'S are eigenvectors of correlation matrices of $\underline{n}$ and $\underline{f}$, respectively, and d_i and a_i are the corresponding coefficients. Then cross-correlations between those coefficients can be written as,

$$E(d_j d_k^*) = \beta_j \delta_{jk} , \tag{12-a}$$

$$E(a_j a_k^*) = \gamma_j \delta_{jk} , \tag{12-b}$$

where δ_{jk} is a Kronecker delta product. Using these relations, Eq.(10) can be rewritten as,

$$\|\underline{r}\|^2 = \sum_{i=1}^{\ell} \lambda_i^{-1} \left| \sum_{j=1}^{M} c_{ij} d_j \right|^2 + \sum_{i=\ell+1}^{N} \left| \sum_{j=1}^{N} b_{ij} a_j \right|^2 , \tag{13-a}$$

$$\text{where} \quad c_{ij} = \underline{u}_i^t \phi_j^M \quad \text{and} \quad b_{ij} = \underline{v}_i^t \phi_j^N \tag{13-b}$$

Taking expectations of Eq.(13-a) gives

$$\begin{aligned} E[\|\underline{r}\|^2] &= \sum_{i=1}^{\ell} \lambda_i^{-1} \sum_{j,m=1}^{M} E(d_j d_m^*) c_{ij} c_{im}^* \\ &+ \sum_{i=\ell+1}^{N} \sum_{j,m=1}^{N} E(a_j a_m^*) b_{ij} b_{im}^* \\ &= \sum_{i=1}^{\ell} \lambda_i^{-1} \sum_{j=1}^{M} \beta_i |c_{ij}|^2 + \sum_{i=\ell+1}^{N} \sum_{j=1}^{N} \gamma_i |b_{ij}|^2 \end{aligned} \tag{14}$$

Now a reasonable termination index ℓ may be determined through iterative computer procedure to minimize $E[\|\underline{r}\|^2]$ under the given signal and noise correlations. A simple formula which unambiguously determines the termination index ℓ may be derived by assuming that $\underline{n}$ and $\underline{f}$ are white so that $\beta_j = S_n$ and $\gamma_j = S_f$ for all j.

With the above assumption, Eq.(14) becomes

$$E[\|\underline{r}\|^2] = M S_n \sum_{i=1}^{\ell} \lambda_i^{-1} + N(N-\ell) S_f , \tag{15}$$

and it has a minimum where

$$\ell = \operatorname*{Max}_{k} \left[k ; \lambda_k \geq \frac{M S_n}{N S_f} \right] , \tag{16}$$

and the λ_k's are ordered in a decreasing manner. Eq.(16) comes from the concave nature of the function $E[\|r(k)\|^2]$ and the following two conditions.

$$\begin{aligned} &E[\|r(\ell)\|^2] - E[\|r(\ell-1)\|^2] \\ &= M S_n \lambda_\ell^{-1} - N S_f \leq 0 \end{aligned} \tag{17-a}$$

$$\begin{aligned} &E[\|r(\ell)\|^2] - E[\|r(\ell+1)\|^2] \\ &= -M S_n \lambda_{\ell+1}^{-1} + N S_f \leq 0 \end{aligned} \tag{17-b}$$

2. Stochastic pseudo-inverse filter

For the derivation of a stochastic pseudo-inverse, the restoration filter equation Eq.(6) may be replaced by

$$[W]_s^+ = \sum_{i=1}^{G} k_i \underline{v}_i \underline{u}_i^t \tag{18}$$

Using the restoration filter of Eq.(18). $E[\|\underline{r}\|^2]$ can be written as,

$$\begin{aligned} E[\|\underline{r}\|^2] &= \sum_{i=1}^{G} [(1-k_i\lambda_i^{1/2})^2 \sum_{j=1}^{N} \gamma_j |b_{ij}|^2 \\ &+ k_i^2 \sum_{j=1}^{M} \beta_i |c_{ij}|^2] \end{aligned} \tag{19}$$

As in the case of truncated filter approach, optimum gain sequence $[k_i]$ can be obtained as,

$$K_i = \frac{\lambda_i^{1/2} \sum_{j=1}^{N} \gamma_j |b_{ij}|^2}{\lambda_i \sum_{j=1}^{N} \gamma_j |b_{ij}|^2 + \sum_{j=1}^{M} \beta_j |c_{ij}|^2} \tag{20}$$

where β_j and γ_j are the same as the previously defined expectations. Also, when $\underline{f}$ and $\underline{n}$ are assumed to be white, the sequence $[k_i]$ is given as,

$$K_i = \frac{\lambda_i^{1/2}}{\lambda_i + (M S_n / N S_f)} \tag{21}$$

Above Eq.(20) and Eq.(21) are in similar forms with the equations given in REF.(7).

IV. Computer Simulation Results

A computer simulation is performed based on the conventional transaxial computerized tomographic reconstruction as shown in Fig. 1. As an alternative to Fourier convolution or iterative technique, by making use of the linear relationship between the original image vector and observed projected data vector, one can make a pseudo-inversion 3-D image reconstruction and restoration.[8]

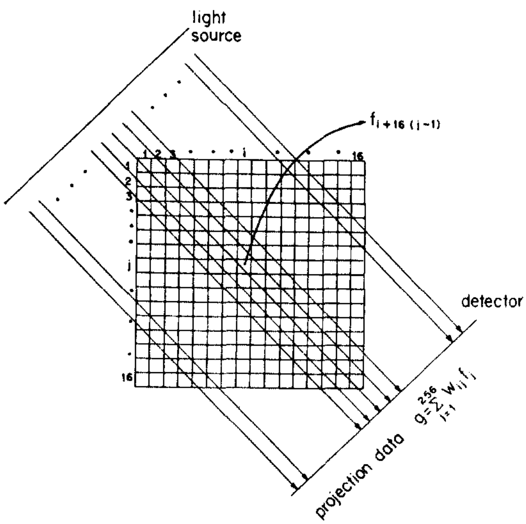
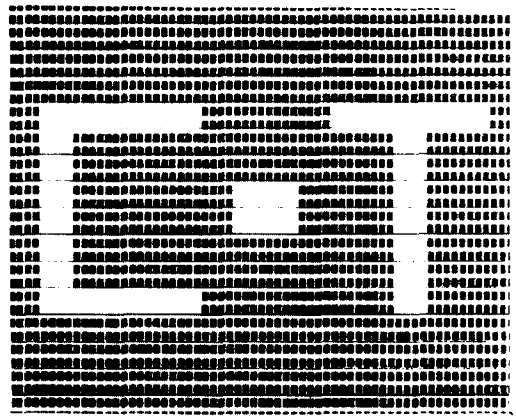


Fig. 1. The linear model of projection operation in computerized tomography.

A simulated phantom consists of characters C·T, sampled with 16x16 point as shown in Fig. 2-(a) in a matrix form through over-printing. Each point inside the characters was given a value 1, each point outside a value 0. This original object blurred by projection operation is shown in Fig. 2-(b). Here projection was performed for 16 beams per each 16 angle, and then a zero mean Gaussian random noise was added to it. A standard deviation of the noise was 0.004.

The restoration was achieved using Eq. (8). Varying termination index ℓ, we obtained $\ell = 1, 2, \ldots, 256$ restored images. Some of them are shown in Fig. 3. These images were also obtained through a line printer by over-printing in 32-grey levels. Fig. 3 shows the effects of noise. For example Fig. 3-(d) is an image which is corrupted by noise amplification. A termination index derived in Eq. (16) gives a good



(a) Simulated phantom

(b) Image blurred by projection

Fig. 2.

restored image. At $\ell = 235$ where $\lambda_\ell^{1/2} = 0.01$, a good restoration is obtained which is in good agreement with the theory presented. (see Fig. 3-(c))

V. Conclusion and Disscussion

It was shown that the SVD approach of designing a restoration system is well suited for the noisy linear degradation by introducing some modification. And truncated pseudo-inverse and stochastic pseudo-inverse filters were designed to minimize the residual error, under the assumption that original image is wide sense staionary. And with further assumption that the image and noise are white, simple formulas were derived in Eq.(16) and Eq.(21). By terminating at an index derived, a good restored image may be obtained. nix

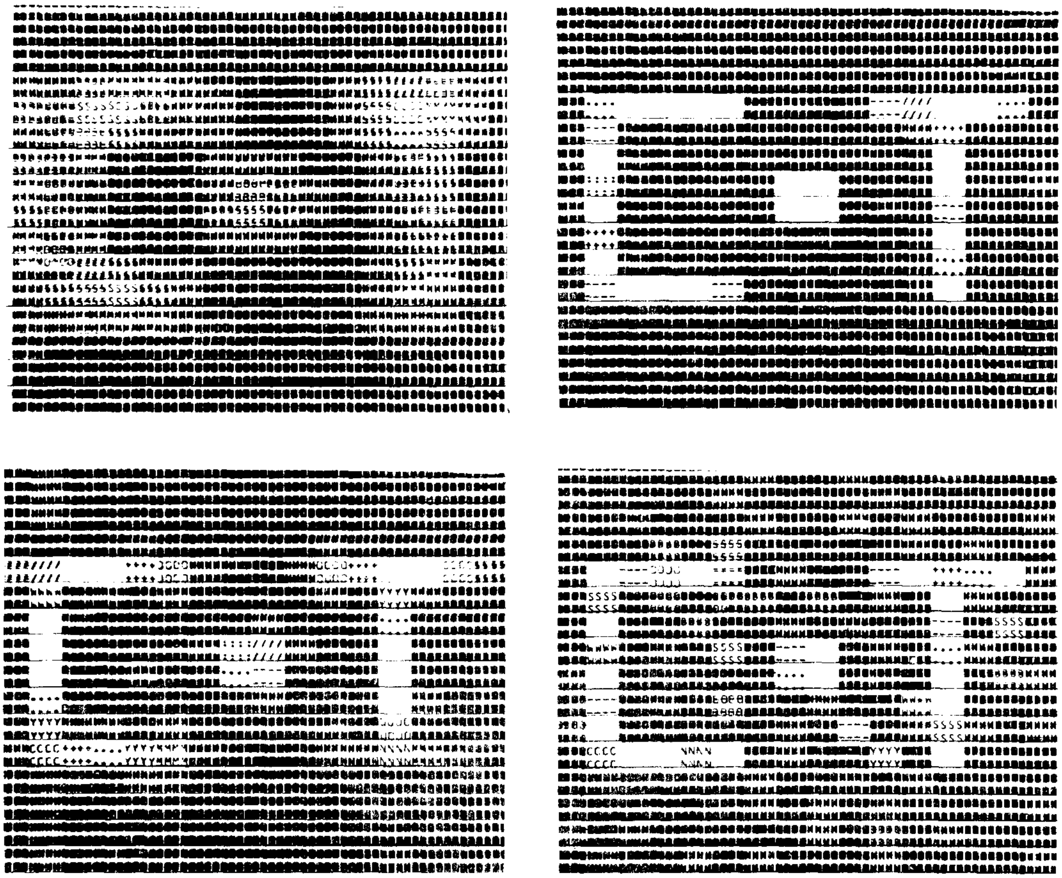


Fig. 3.
(a) Restored image $\ell = 40$
(b) Restored image $\ell = 125$
(c) Restored image $\ell = 235$
(d) Restored image $\ell = 247$

Although the computation of SVD appears complex and requires much more computation time than the other restoration techniques,[9] it should be mentioned that SVD provides an important insight of the restoration problems, since a direct matrix inversion method (the first intuitively tempting method one can think of), is too sensitive to noise to provide a good restoration.

Furthermore, with accurate modeling, one might have an avenue which leads to better reconstructed image in some applications, such as noisy or possibly irrationally scanned image, etc., by use of SVD pseudo-inversion methods.

Appendix

Computer Simulation Programs

Computer simulation was performed in the following procedure with subroutines GWT, GAUS, DISPL, SVA, etc.. First, weighting matrix [W] which represents a linear relation-ship between original image and projection data was computed in subroutine GWT. Its singular value decomposition was carried out by using the subroutines written in the book REF. (6), in which the SVD was performed through successive elementary orthogonal transformations such as Householder and Givens transformations, by adopting quadratic convergence of successive bidiagonal matrices. Finally each restored image corresponding to termination index ℓ was obtained and printed in 16 x 16 matrix form. Here noisy projection data were obtained by weighting matrix [W] and additive Gaussian noise generated in subroutine GAUS.

```
C     SUBROUTINE GWT (MIM.NIM,KBEAM,MANGL,MSIZE2)
C
C     THIS SUBROUTINE COMPUTES THE WEIGHTING MATRIX
C     COEFFICIENTS.
C
C     MIM,NIM ; SIZE OF ORIGINAL IMAGE ARRAY
C     KBEAM   ; NUMBER OF BEAMS PER EACH PROJECTIONS
C     MANGL   ; ANGLE DIFFERENCE BETWEEN THE SUCCESSIVE
C               PROJECTIONS
C     MSIZE2  ; MANGL*KBEAM
C
      SUBROUTINE GWT (MIM,NIM,KBEAM,MANGL,MSIZE2)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON /BLUE/ PWT(256,256)
      INTEGER MTST(20)
C
      F(I,K,M1,M2,T)=-(I-1-M1-(K-M1)/DCOS(T))*DTAN(1.57079632915-T)
     1               +M2+1.
      G(J,K,M1,M2,T)=-FLOAT(J-M2-1)*DTAN(T)+FLOAT(K-M1)/DCOS(T)
     1               +M1+1.
      NP1=NIM+1
      MP1=MIM+1
      M1=NIM/2
      M2=MIM/2
      KSO=KBEAM/2-M1
      PI=3.1415926583
      HPI=1.57079632915
C
      DO 10 L=1,MANGL
      THETA=HPI*FLOAT(L-1)/FLOAT(MANGL)
      DO 20 K=1,KBEAM
      KK=K-KSO-1
      LI=KBEAM*(L-1)+K
      DO 30 I=1,NP1
      DO 40 KLM=1,20
   40 MTST(KLM)=0
      DO 50 J=1,MP1
      LJ=MIM*(I-1)+(NIM-J+1)
      LJ1=MIM*(I-2)+(NIM-J+2)
      IF (L.EQ.1) GO TO 200
C
      FU=F(I,KK+1,M1,M2,THETA)
      FL=F(I,KK,M1,M2,THETA)
      GL=G(J,KK,M1,M2,THETA)
      GR=G(J,KK+1,M1,M2,THETA)
      GR1=G(J+1,KK+1,M1,M2,THETA)
      GL1=G(J-1,KK,M1,M2,THETA)
      FU1=F(I+1,KK+1,M1,M2,THETA)
      FL1=F(I-1,KK,M1,M2,THETA)
C
C
      IF (I.EQ.1.OR.J.EQ.1) N1=1
      IF (I.EQ.NP1.OR.J.EQ.MP1) N2=1
      IF (J.GE.FL.AND.J.LE.FU) MTST(J)=1
      IF (MTST(J).NE.1) GO TO 160
C
      IF (J+1.GE.FU.AND.I+1.GE.GR) GO TO 60
      IF (J+1.GE.FU.AND.I+1.LE.GR) GO TO 70
      IF (J+1.LE.FU.AND.I+1.GE.GR) GO TO 80
      IF (J+1.LE.FU.AND.I+1.LE.GR) GO TO 90
   60 IF (N2.NE.1) PWT(LI,LJ)=(FU-J)*(GR-I)/2.
      GO TO 100
   70 IF (N2.NE.1) PWT(LI,LJ)=(FU+FU1-2.*J)/2.
      GO TO 100
   80 IF (N2.NE.1) PWT(LI,LJ)=(GR+GR1-2.*I)/2.
      GO TO 100
   90 IF (N2.NE.1) PWT(LI,LJ)=1.-(I+1-GR1)*(J+1-FU1)/2.
  100 CONTINUE
C
      IF (I-1.LE.GL.AND.J-1.LE.FL) GO TO 110
      IF (I-1.LE.GL.AND.J-1.GE.FL) GO TO 120
      IF (I-1.GE.GL.AND.J-1.LE.FL) GO TO 130
      IF (I-1.GE.GL.AND.J-1.GE.FL) GO TO 140
  110 IF (N1.NE.1) PWT(LI,LJ1)=(I-GL)*(J-FL)/2.
      GO TO 150
  120 IF (N1.NE.1) PWT(LI,LJ1)=(2.*I-GL-GL1)/2.
      GO TO 150
  130 IF (N1.NE.1) PWT(LI,LJ1)=(2.*J-FL-FL1)/2.
      GO TO 150
  140 IF (N1.NE.1) PWT(LI,LJ1)=1.-(GL1-I+1)*(FL1-J+1)/2.
  150 CONTINUE
C
      IF (J+1.GE.FU.AND.I-1.LE.GL) GO TO 155
      GO TO 160
  155 LJ2=NIM*(I-2)+(NIM-J+1)
      S1=(J+1-FU)**2*DTAN(THETA)/2.
      S2=(GL+1-I)**2*DTAN(HPI-THETA)/2.
      AIR=1-S1-S2
      IF (I.NE.1.AND.J.NE.NP1) PWT(LI,LJ2)=AIR
      GO TO 160
C
  200 IF (I.EQ.K-KSO.AND.I.NE.NP1) GO TO 220
      GO TO 160
  220 IF (J.NE.NP1) PWT(LI,LJ)=1.
  160 N1=0
      N2=0
   50 CONTINUE
   30 CONTINUE
   20 CONTINUE
   10 CONTINUE
C
      DO 230 LI=1,MSIZE2
      DO 230 I=1,NIM
      DO 230 J=1,MIM
      KI=NIM+1-J
      KJ=I
      LJ1=MIM*(I-1)+(NIM+1-J)
      LJ2=MIM*(KI-1)+(NIM+1-KJ)
      PWT(LI+MSIZE2,LJ2)=PWT(LI,LJ1)
  230 CONTINUE
C
      RETURN
      END
```

We have listed subroutine GWT which computes the matrix coefficients effectively. Other subroutines DISPL used for grey scale over-printing and GAUS are popular. And the subroutines which perform singular value decomposition, SVA, SVDRS, QRBD, H12, G1, G2, DIFF can be found in the book REF. (6).

References

1. H. Andrew, "Two-Dimensional Transforms," in Digital Picture Processing, T.S. Huang, Ed., Springer, Berlin, 1975.
2. H. Andrews and C. Patterson, "Outer Product Expansions and Their Uses in Digital Image Processing," IEEE Trans. Computers, C-25, No. 2, pp. 140-148, Feb. 1976.
3. Topics in Applied physics, "Picture Processing and Digital Filtering", edited by T.S. Huang, Springer Verlag New York pp. 6-9, 1969.
4. Z.H. Cho, J. Chan, E.L. Hall, R.P. Kruger, and D.G. McGauhey, "A Comparative Study of 3-D Image Reconstruction Algorithms with Reference to Number of Projections and Noise Filtering," IEEE Trans. Nucl. Sci. NS-22, pp. 344-358, Feb. 1975.
5. Z.H. Cho and J. Burger, "Construction, Restoration, and Enhancement of 2 and 3-Dimensional Images," IEEE Trans. Nucl. Sci., NS-24, No. 2, pp. 886-899, Apr. 1977.
6. Lawson and Hanson, *Solving Least Squares Problems,* Englewood Cliffs, N.J.; Prentice-Hall, Inc., 1974.
7. W.K. Pratt, "Generalized Wiener Filtering Computation Techniques," IEEE Trans. Computers, C-21, No. 7 pp. 636-641, July 1972.
8. Kwoi-Kai Chan, "Transverse Axial Positron Camera," Ph.D. Thesis in UCLA, 1976.
9. H.C. Andrews, B.R. Hunt, *Digital Image Restoration,* Englewood Cliffs, N.J.; Prentice-Hall, Inc., 1977.