Critical Review of Reconstruction Filters for Convolution Algorithms

(Convolution 알고리즘을 이용한 映像 再構成 필터에 관한 硏究)

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要 約

三次元 映像 再構成을 위해서는 Fourier convolution 알고리즘이 주로 使用되고 있다. 이때 사용되는 필터의 設計에는 몇가지 選擇의 餘地가 있다.

이 論文은 지금까지 提案된 映像 再構成 필터들을 sampling rate와 aliasing과 雜音의 見地에서 理論的으로 比較, 檢討하고 이를 土臺로 새로운 필터의 設計方法을 提案하였다.

또한 컴퓨터 simulation을 通해 旣存의 再構成 필터와 提案된 필터의 理論的인 分析 結果를 確認,檢討하였다.

Abstract

The Fourier convolution algorithm is used to reconstruct a 3-D density function from projection data sets. The convolved data are then back-projected to obtain a density function. There are several choices of the weighting function for the design of the reconstruction(deblurring) filter.

Present paper reviews the published reconstruction filters theoretically and proposes a new reconstruction filter design method, considering the problems such as the effects of sampling rate, aliasing, and noise.

Several previous reconstruction filters are compared with the proposed filter by computer simulations.

1. Introduction

Computerized tomography (CT) is a radiographic technique that provides an image of a two dimensional slice of a three-dimensional object. Since the three-dimensional object can be considered as the stack of the two dimensional slices, CT is used to determine the density function inside the object. This technique brught a great advancement in the medical diagnosis which remained for nearly a century with the ordinary two-dimensional X-ray technique.

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The fundamental problem of the CT is the reconstruction of the section image from one dimensional projection data which are obtained by the X-ray projections through the object slice at many different angles. A simple reconstruction method using the collected angle projection data is simply to back-project the projection data into a common image plane and add them. According to this method, the point image is formed by superposition of straight lines, hence the image obtained is blurred by the function proportional to $1/r$ where r is a distance from a image point.[1]

For obtaining more accurate image, many reconstruction methods have been proposed in the recent years.[2-5] Among them, the Fourier convolution method is widely used due to its simplicity, computational efficiency, and high immunity to noise.

The Fourier transform method of reconstruction of the 2-dimensional density function $\mu(x,y)$ is represented by the following equation.

$$\mu(x,y) = \frac{1}{(2\pi)^2}\int_0^{\pi} d\theta \int_{-\infty}^{\infty} dw\, |w|\, P(w,\theta) \cdot \exp(iw(x\cos\theta + y\sin\theta)) \quad \cdots\cdots (1)$$

where w is the spatial frequency and $P(w,\theta)$ is the Fourier transform of the projection data $p_\theta(R)$. If $h(R)$ is the inverse Fourier transform of $H(w) = |w|$, the equation (1) can be rewritten in convolution form:

$$\mu(x,y) = \frac{1}{2\pi}\int_0^{\pi} d\theta \int_{-\infty}^{\infty} h(R-R')\, p_\theta(R')\, dR' \quad \cdots\cdots (2)$$

where $R'=x\cos\theta+y\sin\theta = r\cos(\theta-\phi)$. Note that $\mu(x,y)$ is obtained by summation or back-projection of the convolved projection with weighting function $h(R)$. Since the projection data are practically obtained only for $R=R_k=ka$ $(k=0, \pm1, \pm2, \cdots)$, and $\theta=\theta_j=j\pi/n$ $(j=0, 1, \cdots\cdots, n-1)$ where a is a sampling space in each view and n is the number of views, the equation (2) may be approximated by,

$$\mu(x,y) = \frac{a}{2n}\sum_{j=0}^{n-1}\sum_{k=1}^{m} p_{\theta_j}(R_k)\, h(R-R_k) \quad \cdots\cdots (3)$$

Here, m is the number of sampling points of the projection data, and is decided by the size of object plane.

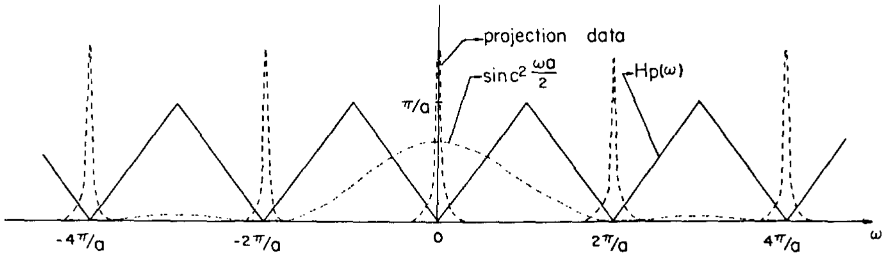
The remaining problem for the reconstruction is to set the weighting function $h(R)$, which does not really exist because its Fourier transform $|w|$ diverges as w goes to infinity. If $|w|$ is replaced with the function which approximates $|w|$ for $|w| < w_c$ and smoothly goes to zero for $|w| > w_c$, $h(R)$ is realizable. This approximation is applicable because $P_\theta(w)$ has very small amount of signal power at $|w| > w_c$, therefore very low signal to noise ratio.

This condition for the realization of $h(R)$ can be satisfied from another approach by setting the latter summation part of equation (3) to $Q(R)$,

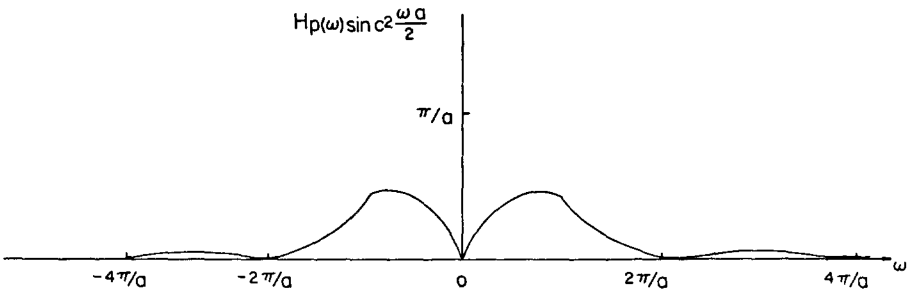
$$Q(R) = \sum_{k=1}^{m} p_{\theta_j}(R_k)\; h(R-R_k) \quad \cdots\cdots (4)$$

$Q(R)$ has to be calculated for every point (x, y) where the density function is to be mapped, because $h(R)$ is a smooth and continuous function. This approach is obviously a costly approach. If we assume, however, that $h(R)$ is a piecewise linear function with the interval of a, all the values of $Q(R)$ can be calculated with much smaller computation time by linear interpolation with only m values of $Q(R_i)$, which are the digital convolution of sampled projection data and delta array of $h(R)$ with sampling space of a. By sampling theorem, this process represents the product of the pen periodic frequency spectrum of projection data and that of sampled $h(R)$ which is weighted with sinc^2 function. The latter is due to the linear interpolation in the spatial domain. (Fig. 1) Thus, the product of sinc^2 and the periodic spectrum of sampled $h(R)$, which is to be called $H_p(w)$, can be considered as the resultant reconstruction filter which is approximately $|w|$ for low frequency and goes to zero smoothly for high frequency. In other words, the filter function $H_p(w)$ can be modified so that $H_p(w)$ is approximately equal to $|w|$ in the vicinity of $2n\pi/a$, and can be adjusted for $2n\pi/a < w < 2(n+1)\pi/a$ according to the system characteristic.

The filter shown in Fig. 1 is same as the one proposed by Ramachandran and Lakshiminaryanan.[4] Shepp and Logan suggested modified filter which is an absolute sine wave.[5] In 1977, Kwoh et al. proposed a generalized filter[6] where $H_p(w)$ in the intervals of $2n\pi/a$ and $2(n+1)\pi/a$ can adjusted in accordance with the system noise.



(a) Frequency spectra of a sampled weighting function, projection data, and linear interpolation function.



(b) A resultant reconstruction filter.

Fig. 1.

The objectives of this paper are to examine the above filters and to consider a more improved filter design method from the viewpoint of noise, resolution, and aliasing effect for the given system.

II. Review of Several Reconstruction Filters

As was discussed above, main problem of the reconstruction filter design is how to weight the periodic function $H_p(w)$ with the constraint so that it is appoximately $|w|$ in the region adjacent to $2n\pi/a$. In the following discussion, $H_p(w)$ in the range from 0 to $2\pi/a$ will be considered because it is a periodic function with period $2\pi/a$.

1. Ramachandran and Lakshiminarayanan filter

A filter defined by Ramachandran and Lakshiminarayanan is,

$$H_p(w) = \begin{cases} w & \text{for } 0 < w \leq \pi/a \\ \dfrac{2\pi}{a} - w & \text{for } \pi/a < w < 2\pi/a \end{cases} \quad \cdots\cdots (5)$$

and the spatial impulse response of the filter, h(R), the inverse Fourier transform of $H_p(w)$ was determined as,

$$h(0) = \frac{\pi}{2a^2}$$

$$h(ka) = \begin{cases} -\dfrac{2}{\pi k^2 a^2} & \text{for k odd integer} \\ 0 & \text{for k even integer} \end{cases} \quad \cdots\cdots (6)$$

For low noise, the reconstructed image by this filter has good resolution, while it is somewhat oscillatory due to the fact that $H_p(w)$ is not smooth at $w = \frac{\pi}{a}$.

2. Shepp and Logan filter

Shepp and Logan modified the above filter and suggested following filter,

$$H_p(w) = \frac{2}{a}\left|\sin\frac{wa}{2}\right|$$

$$h(ka) = -\frac{4}{\pi a^2(4k^2-1)} \quad \cdots\cdots (7)$$

The obtained image, still preserving the accuracy needed, is less oscillatory and less sensitive to noise than Ram & Lak filter. These results come from the fact that the function becomes smooth at $w=\pi/a$, and the filter energy, which is proportional to the reconstruction image variance corrupted by noise,[5] is somewhat decreased.

3. Generalized $|w|$-filter

While both Ram & Lak and Shepp & Logan filters are generally satisfactory for low noise image reconstruction, they suffer image degradation at high noise levels. In 1977, Kwoh, Reed, and Troung proposed a generalezed $|w|$-filter applicable to the image reconstruction with wide range of noise levels.[6] The suggested $H_p(w)$ is,

$$H_p(w) = \begin{cases} w\, e^{-\xi w^p} & 0 < w < \pi/a \\ \left(\frac{2\pi}{a} - w\right) e^{-\xi\left(\frac{2\pi}{a} - w\right)^p} & \frac{\pi}{a} \leq w < \frac{2\pi}{a} \end{cases} \quad \cdots\cdots (8)$$

In this filter, $e^{-\xi w^p}$ and $e^{-\xi\left(\frac{2\pi}{a} - w\right)^p}$ reduce the filter energy of Ram & Lak filter to improve the noise filtering. On the other hand, the reduction of the filter energy results in resolution degradation because of decreases of high frequency terms of the signal. Thus, the design factor, ξ and p are to be adjusted in ad hoc basis to the adequate values for a given signal to noise of the image to be reconstructed.

Ⅲ. New Reconstruction Filter Design Method

In the above generalized filter, numerical analysis must be used to obtain the impulse response h(R) for the conventional convolution reconstruction. It is also required to choose ξ and p so that $H_p(w)$ is smooth at π/a. A new, simple and versatile filter is, therefore, proposed to solve these problems. The proposed new scheme is as follows;

$$H_p(w) = \frac{2}{a} \left| \sin \frac{wa}{2} \right| X(w) \quad \cdots\cdots (9)$$

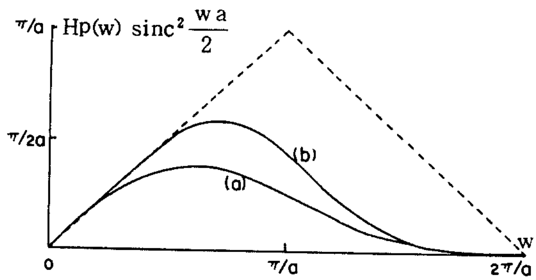
where $X(w) = p + q \cos wa + r \cos 2wa$ and $p + q + r = 1$

As shown in Appendix, the Fourier inversion formula then gives.

$$h(ka) = -\frac{2}{\pi a^2}\left(\frac{2p-q}{4k^2-1} + \frac{3(q-r)}{4k^2-9} + \frac{5r}{4k^2-25}\right) \quad \cdots\cdots (10)$$

Here p, q, and r are determined by the given system noise. This filter is smooth at π/a for any p, q, and r, and its inverse Fourier transform can be obtained easily in comparison with the generalized $|w|$-filter. If we increase the high frequency cosine terms, it is equivalent to an increase of Fourier terms. Thus, if the sampled projection data and noise spectrum, and aliasing effect are estimated, the coefficients of $X(w)$ can be determined for the optimum filter.

While the generalized filter is concerned only with the noise, the proposed filter can be adjusted properly for both noise degradation at high noise and the resolution improvement at very low noise situation. For examples of the latter case, we can think of the non-destructive testing system, which permits large X-ray dosage, and the system with low sampling rate so that the signal al to noise ratio is very large. In these cases, we can obtain the higher resolution for the given system condition by increasing the filter



(a) Shepp's filter
(b) A filter for resolution improvement, considering aliasing effect

Fig. 2. The frequency spectra of two reconstruction filters.

energy with adequate p, q, and r. It is reason - able because the Spepp's filter, as an example, $H_p(w)\ \mathrm{sinc}^2 \frac{aw}{2}$ is different from the ideal $|w|$ filter in the medium frequency range, as shown in Fig. 2.

IV. Simulations and Discussions

The responses of Ram and Lak filter, Shepp and Logan filter, and the porposed filter to a circular phantom are simulated in order to confirm the above theoretical comparison for the previous two existing reconstruction filters and the proposed filter. As shown in Fig. 3, the phantom consists of two large circles and nine small circles of the density difference within 2%

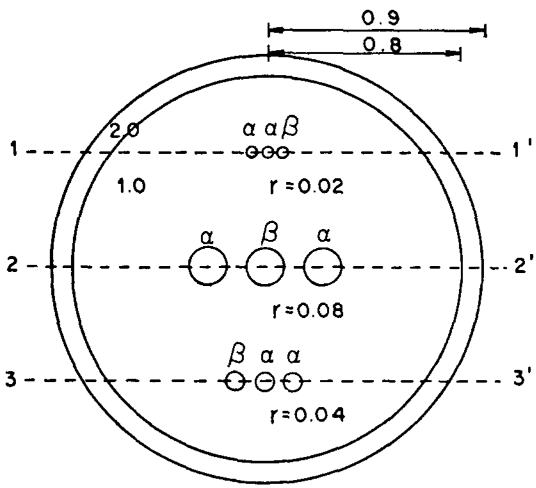
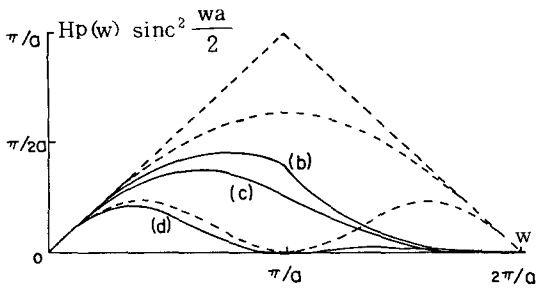


Fig. 3 The phantom for simulation. ($\alpha = 1.01$, $\beta = 1.02$)

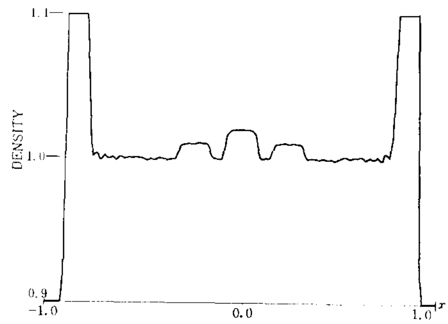
50 views, each of which has projection data of 100 points, are used for reconstruction. The reconstruction is performed at only three lines which pass through the 3 circles, respectively.

Fig. 4 shows the noiseless results of three filters. As was discussed, Ram & Lak filter is somewhat oscillatory and the resolution is low - ered as the filter energy decrease.

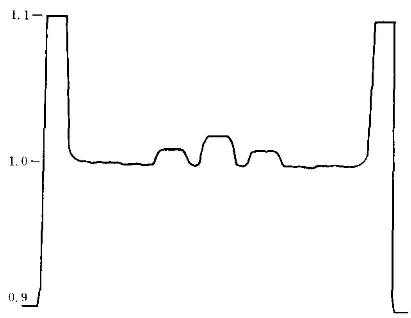
It is also noted in Fig. 5 that the resolution is improved as the filter energy increases. The undershoot in Fig.5(c) is considered as the result



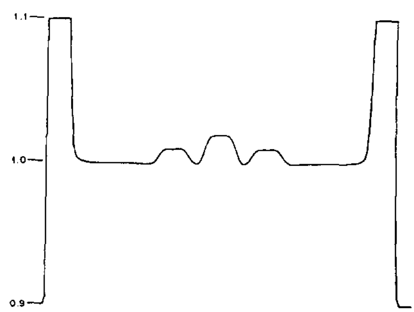
(a) Filter shapes



(b) Ram & Lak



(c) Shepp & Logan (p = 1, q = 0, r = 0)



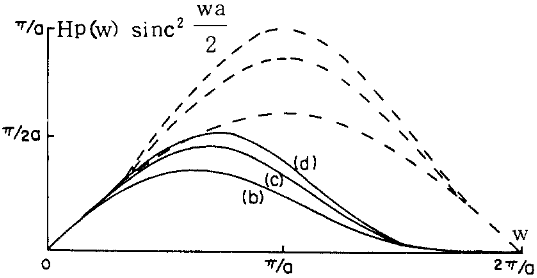
(d) The proposed filter (p = 0.5, q = 0.5, r = 0)

Fig. 4. Cut views at the line 2-2′ with no noise.

of aliasing effect which can be compensated.

Two levels of noise are chosen to demonstrate the reconstruction in noise. Fig. 6 shows the simulation results which were expected theoretically. It is shown that the filter with lower filter energy is required for the satisfactory result when noise standard deviation σ equals 0.005.

Since high frequency components of the projection data are damped for large noise case, high

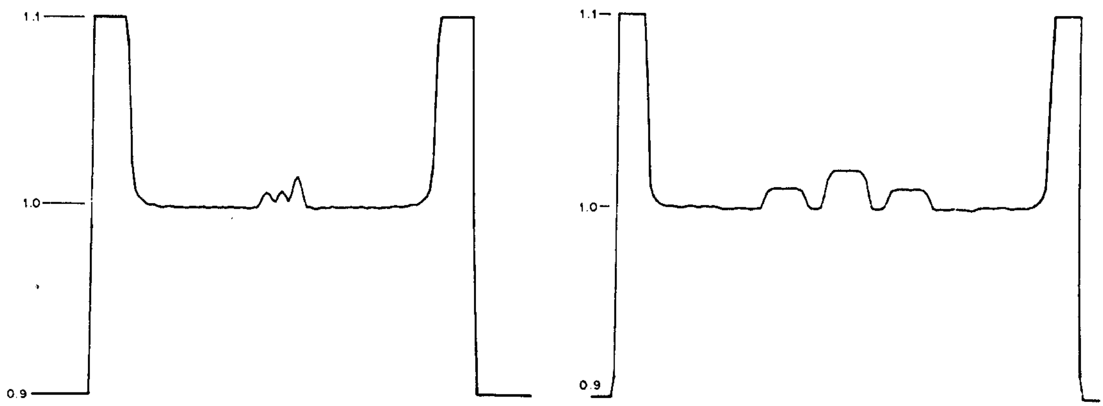


(a) Filter shapes

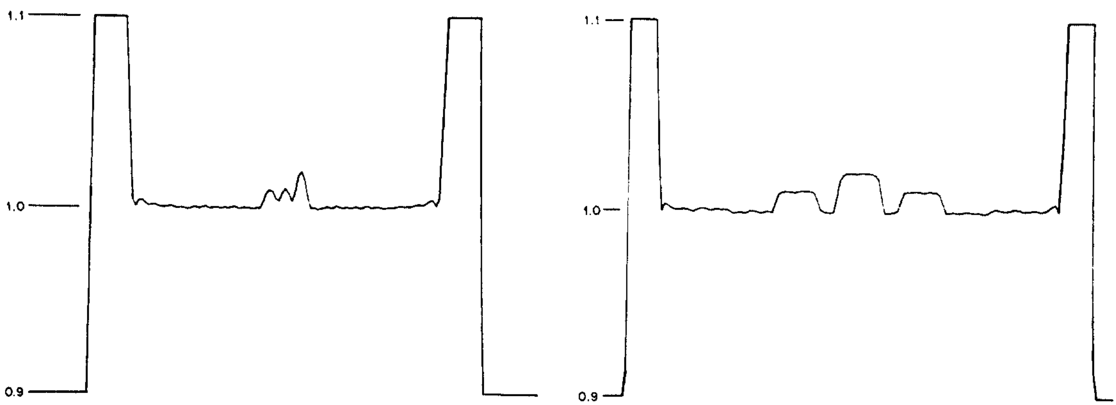
sampling rate may not make a contribution to the resolution improvement except reducing the aliasing effect. This effect is seen by comparison of the results shown in Fig.6(d) and Fig.7.

This means that number of detectors can be reduced without sacrificing the system performance if noise is large.

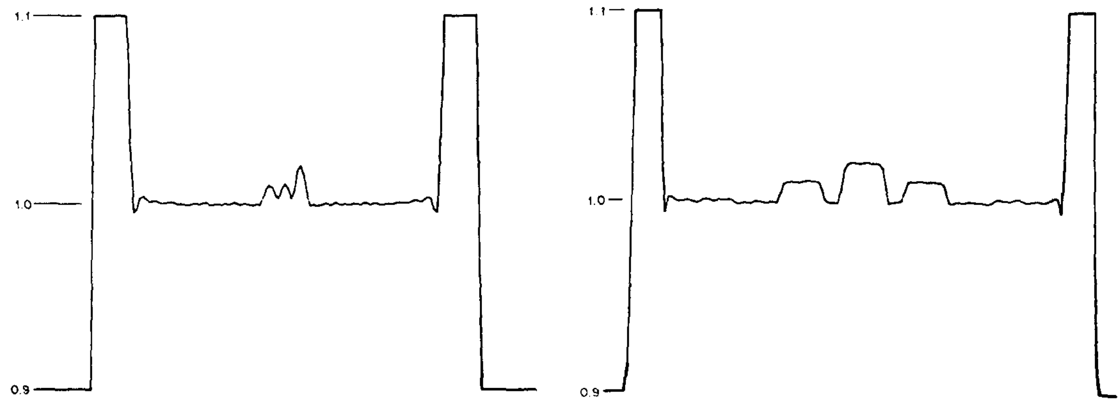
Previously proposed filters are analyzed theoretically from the viewpoint of the noise, resolution, and aliasing effect. From these theoretical bases, a new design method of the reconstruction filter is proposed. This method produces better results when the system characteristics, namely, noise, and number of detectors, etc., are given. It can also be incorporated with an approximated Wiener filter design which is sometimes simple and easy to handle.



(b) $p=1$, $q=0$, $r=0$

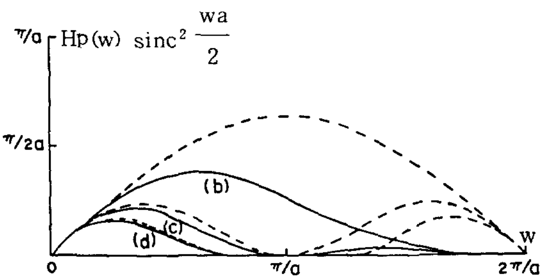


(c) $p=1.2$, $q=-0.2$, $r=0$

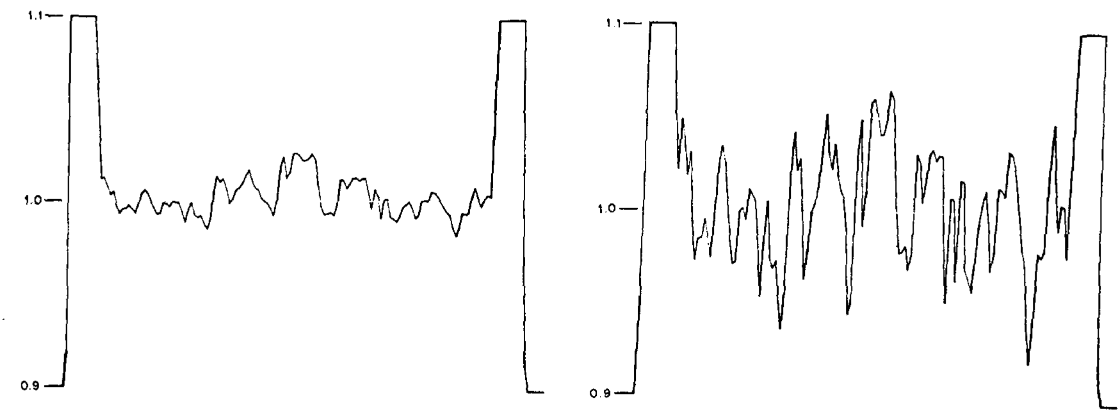


(d) p = 1.3, q = − 0.3, r = 0

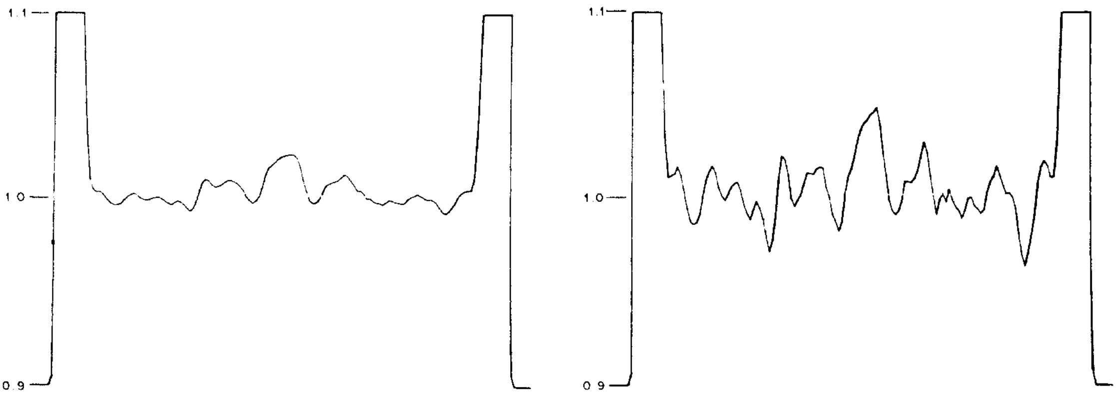
Fig. 5. Cut views at the lines, 1− 1′ and 2−2, with no noise.



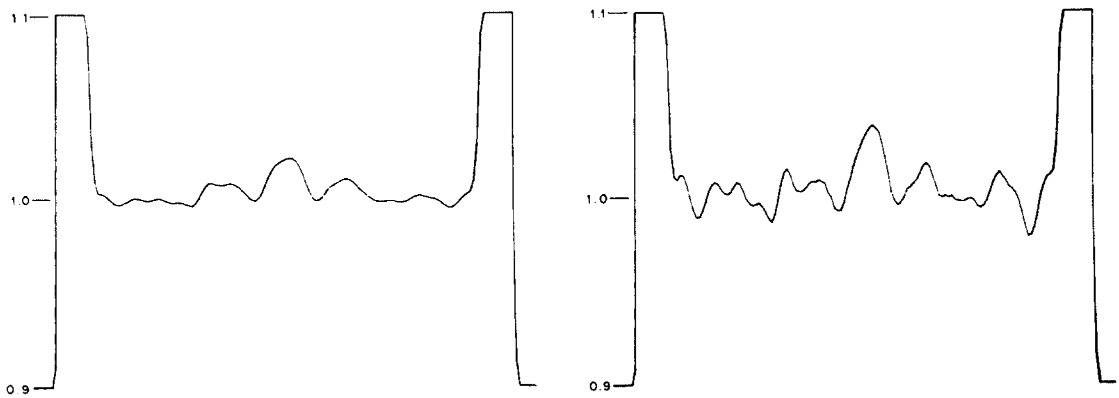
(a) Filter shapes



(b) p = 1, q = 0, r = 0

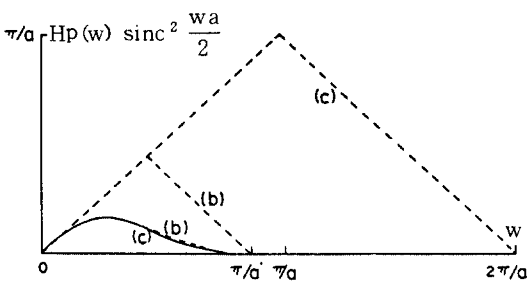


(c) p = 0.5 , q = 0.5 , r = 0



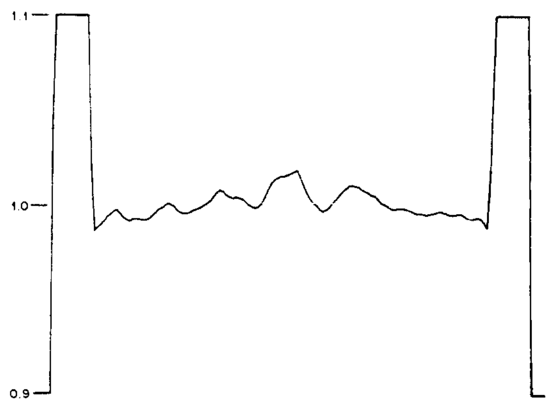
(d) p = 0.35 , q = 0.5 , r = 0.15

Fig. 6. Cut views, at the line 2-2′, of the reconstructed image with noise.



a′ = 2/44   a = 2/100

(a) Shape of Shepp & Logan filter for 44 sampling points in comparison with the proposed filter (p = 0.35, q = 0.5, r = 0.15) for 100 sampling points. The filter of (c) is the same filter as Fig.6(d).



(b) Result of the reconstruction with noise ($\sigma = 0.1\,\%$)

Fig. 7. Simulation with the projection data of 44 sampling points at the line 2-2′.

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Appendix

If we consider the only one period of $H_p(w)$, its inverse Fourier transform is as follows.

$$h(x)=\frac{1}{2\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} H_P(w)\, e^{jxw}\, dw$$

$$=\frac{1}{\pi}\int_0^{\frac{\pi}{a}} H_P(w)\cos xw\; dw \cdots\cdots (A.1)$$

where

$$H_P(w)=\frac{2}{a}\left|\sin\frac{wa}{2}\right|(p+q\cos wa+r\cos 2wa$$

$$\cdots\cdots (A.2)$$

By sampling theorem, the inverse Fourier transform of the periodic function $H_P(w)$ is

$$F^{-1}[H_P(w)]=h(X)\sum_{k=-\infty}^{\infty}\delta(x-ka)$$

$$=\sum_{k=-\infty}^{\infty} h(ka)\cdots\cdots (A.3)$$

From (A.1) and (A.2),

$$h(ka)=h(x)\,|_{x=ka}$$

$$=\frac{1}{\pi}\int_0^{\frac{\pi}{a}}\frac{2}{a}\sin\frac{wa}{2}(p+q\cos wa+r\cos 2wa)$$

$$\cdot \cos kaw\; dw$$

$$=\frac{1}{2\pi a}\int_0^{\frac{\pi}{a}}[(2p-q)(\sin\frac{2k+1}{2}wa$$

$$-\sin\frac{2k-1}{2}wa)+(q-r)(\sin\frac{2k+3}{2}wa$$

$$-\sin\frac{2k-3}{2}wa)+r(\sin\frac{2k+5}{2}wa$$

$$-\sin\frac{2k-5}{2}wa)\; dw$$

$$=-\frac{2}{\pi a^2}\left(\frac{2p-q}{4k^2-1}+\frac{3(q-r)}{4k^2-9}+\frac{5r}{4k^2-25}\right)$$

$$\cdots\cdots (A.4)$$

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