不規則 Pulse 信号의 電力 Spectrum
(Power Spectrum of Jittered Random Pulse Train)

崔 陽 熙* 金 在 均**
(Choi, Yang Hee and Kim, Jae Kyoon)

要 約

位相흔들림이 있는 펄스信號의 電力 스펙트럼을 구하였다. 여기서 pulse sequence는 統計的으로 independent stationary 이며 位相흔들림은 stationary Gaussian의 確率特性을 갖는 것으로 가정되었다.

Unipolar pulse 信號에 있어서, 位相흔들림이 增加할 때에 電力 스펙트럼의 discrete 部分은 減少하지만 continuous 部分은 增加하는 것으로 나타났다.

Abstract

The power spectrum of jittered pulse train is derived for the independent stationary pulse sequence with a stationary Gaussian phase jitter.

For the unipolar pulse train signal, it is shown that as the phase jitter increases, the continuous part of the power spectrum increases while the discrete part decreases.

1. Introduction

Bennett had shown that in general there are both discrete and continuous components of power spectrum in the random pulse train[1]. The discrete power component at the pulse rate is conveniently used for the time synchronization in many communication systems[2,3]. But the received and regenerated pulse sequence has usually some phase jitter due to the channel noise and other irreqularities.

We have derived the power spectrum of the jittered random pulse train, which is an extention of the Bennett's result. It is assumed that the pulse train is an independent stationary sequence with an independent stationary Gaussian jitter.

* 正會員, 韓國通信技術硏究所
(Korea Telecommunications Research Institute)
** 正會員, 韓國科學院 電氣 및 電子工學科
(Dept. of Electrical Science, KAIS)

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It is shown that both the discrete and the continuous power spectral components are significantly affected by the phase jitter.

2. Power Spectrum

The jittered random pulse train is represented as following.

$$x(t) = \sum_{m=-\infty}^{\infty} a_m\, g(t - mT - \delta_m) \quad \cdots\cdots\cdots\cdots (1)$$

where a_m is a random variable witch binary values 1 or 0. $g(t)$ is the waveform of a single pulse, T is the period of the pulse train, and δ_m is the phase jitter in the m^{th} pulse. In the above representation, both $\{a_m\}$ and $\{\delta_m\}$ are assumed to be independent wide-sense stationary random sequences and also mutually independent. Moreover, δ_m is assumed to be zero-mean Gaussian.

The random signal $x(t)$ may be decomposed to an average signal $E[x(t)]$ and a zero-mean random signal $y(t) = x(t) - E[x(t)]$. We will find the power spectral components corresponding to these two signal components, similarly to Bennett's approach[1]. First the statistical average signal is evaluated.

$$E[x(t)] = \sum_m E[a_m] \cdot E[g(t-mT-\delta_m)]$$

$$= p\sum_m \int_{-\infty}^{\infty} g(t-mT-\delta_m)\, p'(\delta_m)d\delta_m$$

$$= p\sum_m \int_{-T/2}^{T/2} g(t-mT-\delta)p(\delta)d\delta \cdots (2)$$

where $p = \text{prob}(a_m = 1)$, and $p'(\delta_m)$ is the probability density of the stationary phase jitter δ_m. When the jitter is larger than $T/2$, it is better treated as a jitter from the adjacent pulse position. Hence $p'(\delta)$ is folded over the interval of $|\delta| < T/2$, and it is represented by a modified density $p(\delta)$, which is used in (2). Since $E[x(t)]$ is a periodic function of time with the period T, it is expandable to a Fourier series.

$$E[x(t)] = \sum_{k=-\infty}^{\infty} C_k \exp(j\,2\pi k f_c t) \cdots\cdots\cdots (3)$$

where $f_c = T^{-1}$, and the coefficients are

$$C_k = f_c \cdot \int_0^T E[x(t)] \cdot \exp(-j\,2\pi k f_c t)dt$$

$$= pf_c \cdot \int_{-T/2}^{T/2} p(\delta)\sum_m \int_{-mT}^{-(m-1)T} g(u-\delta)\exp(-j2\pi k f_c u)du\, d\delta$$

$$= pf_c \int_{-T/2}^{T/2} p(\delta)\,G(kf_c)\exp(-j\,2\pi k f_c \delta)d\delta$$

$$= pf_c\, G(kf_c)\, D(kf_c) \cdots\cdots\cdots (4)$$

where $D(f)$ and $G(t)$ are Fourier transforms of $p(\delta)$ and $g(t)$, respectively. Notice that $D(f)$ is real and even when $p(\delta)$ is real and even, which is true in general. From (4), the discrete power spectral components are obtained for $f = kf_c$.

$$S_x(kf_c) = p^2 f_c^2\, |G(kf_c)|^2 |D(kf_c)|^2, k=0, \pm 1, \cdots (5)$$

Now the continuous part of the power spectrum will be calculated from the zero-mean random signal $y(t)$. First, its truncated signal $y_N(t)$ is defined as following.

$$y(t) = x(t) - E[x(t)]$$

$$y_N(t) \triangleq \sum_{m=-N}^{N} a_m g(t-mT-\delta_m) - p\sum_{m=-N}^{N} \int_{-T/2}^{T/2} g(t-mT-\delta_m)\, p(\delta_m)\, d\delta_m$$

Then, the Fourier transform of $y_N(t)$ is given by

$$Y_N(f) = \sum_{m=-N}^{N} G(f)\exp(-j\omega mT)\{a_m \exp(-j\omega\delta_m) - pD(f)\}$$

and

$$E[|Y_N(f)|^2] = \sum_{m=-N}^{N}\sum_{n=-N}^{N} |G(f)|^2 \exp[j\omega T(n-m)] \cdot \{E[a_m a_n]E[\exp\{j\omega(\delta_n-\delta_m)\}] + p^2 D^2(f) - 2p^2 D(f) \cdot E[\exp(j\omega\delta_n)]\}$$

$$\cdots\cdots\cdots (6)$$

where $\omega = 2\pi f$, and $E[a_m a_n] = R_a(m-n) = R_a(k) = \begin{bmatrix} p & k=0 \\ p^2 & k \neq 0 \end{bmatrix}, k = |m-n|$.

We notice that for independent and stationary Gaussian sequence $\{\delta_m\}$, the variance of $\delta_n - \delta_m$ is

$$\sigma^2_{n-m} \triangleq \begin{bmatrix} 2\sigma^2 & n \neq m \\ 0 & n = m \end{bmatrix} \cdots\cdots\cdots (7)$$

and $E[\exp\{j\omega(\delta_n-\delta_m)\}] = \exp(-\frac{1}{2}\omega^2\sigma^2)$, where σ^2 is the variance of δ_m. Using (7), and with some change of variable [1], (6) is simplified as following.

$$E[|Y_N(f)|^2] = (2N+1)\,|G(f)|^2\{R_a(0) + \Delta(f) + 2\sum_{k=1}^{2N}(1-\frac{k}{2N+1}) \cdot [R_a(k)\exp(-\omega^2\sigma^2) + \Delta(f)]\cos\omega kT\}, \cdots\cdots (8)$$

where

$$\Delta(f) \triangleq p^2\{D^2(f) - 2D(f)\exp(-\frac{1}{2}\omega^2\sigma^2)\}$$

$$\cdots\cdots\cdots (9)$$

Then, by the definition of power spectrum, the continuous power spectrum is obtained as a limit of the normalized mean squared Fourier transform.

$$S_y(f) = \lim_{N\to\infty} \frac{E[|Y_N(f)|^2]}{(2N+1)T}$$

$$=f_c|G(f)|^2\{R_a(0)+\Delta(f)+2\sum_{k=1}^{\infty}[R_a(k)\exp(-\omega^2\sigma^2)+\Delta(f)]\cos\omega kT\}$$

$$=f_c|G(f)|^2\{p+\Delta(f)+2\sum_{k=1}^{\infty}[p^2\exp(-\omega^2\sigma^2)+\Delta(f)]\cos\omega kT\} \cdots\cdots (10)$$

Assuming that the density function $p(\delta)$ is approximately Gaussian with zero-mean and variance σ^2, the Fourier transform of $p(\delta)$ is approximated as following.

$$D(f)\simeq\exp(-\frac{1}{2}\omega^2\sigma^2) \cdots\cdots (11)$$

Thus, from (9) and (10), the continuous power spectrum $S_y(f)$ is given by

$$S_y(f)=f_c|G(f)|^2p[1-p\exp(-\omega^2\sigma^2)] \cdots\cdots (12)$$

Finally, combining the two spectral components of (5) and (12), the power spectrum $S_x(f)$ of the random pulse signal with independent jitter is expressed as following.

$$S_x(f)=S_y(f)+\sum_{k=-\infty}^{\infty}S_x(kf_c)\delta(f-kf_c)$$

$$=f_c|G(f)|^2p[1-p\exp(-\omega^2\sigma^2)]+\sum_{k=-\infty}^{\infty}p^2f_c^2|G(kf_c)|^2\cdot|D(kf_c)|^2\delta(f-kf_c) \cdots\cdots (13)$$

When there is not any jitter, i.e., in the case of $D(f)=1$, $\sigma^2=0$, the above equation becomes same as the result derived by Bennett [1].

$$S_x(f)=f_c|G(f)|^2p(1-p)+p^2f_c^2\sum_{k=-\infty}^{\infty}|G(kf_c)|^2\delta(f-kf_c) \cdots\cdots (14)$$

We notice that the assumption of independence on $\{\delta_m\}$ is applied only for equation (7). The case of dependent Gaussian jitter can be similarly analyzed by the same procedures [4].

3. Spectrum of Unipolar Pulse Train

For example, to see the effect of the phase jitter, we examine the spectrum for a jittered random pulse-train signal, which has a unit-height rectangular pulse $g(t)$ with pulse width $\frac{T}{R}$ second and an equallikely binary random variable $\{a_m\}$. In this random signal $x(t)$, the Fourier transform of $g(t)$ is

$$G(f)=\frac{T}{R}\operatorname{sinc}(\frac{T}{R}f)=\frac{1}{Rf_c}\operatorname{sinc}(\frac{f}{Rf_c}) \cdots\cdots (15)$$

and from (13), the power spectrum is given by

$$S_x(f)=\frac{1}{R^2f_c}\operatorname{sinc}^2(\frac{f}{Rf_c})\frac{1}{2}[1-\frac{1}{2}\exp(-\omega^2\sigma^2)]+\frac{1}{4R^2}\sum_{k=-\infty}^{\infty}\operatorname{sinc}^2(\frac{k}{R})\exp(-k\omega_c{}^2\sigma^2)\delta(f-kf_c). \cdots\cdots (16)$$

For convenience, we describe the jitter variance σ^2 as a fraction of pulse interval,

$$\sigma=\frac{1}{b}\ (\text{radian})=\frac{1}{b}\frac{T}{2\pi}\ (\text{sec})=\frac{1}{b\omega_c}\ (\text{sec})$$

Then (16) is expressed as the following.

$$S_x(f)=\frac{1}{2R^2\cdot f_c}\cdot\operatorname{sinc}^2(\frac{f}{Rf_c})\{1-\frac{1}{2}\exp[-(\frac{\omega}{b\omega_c})^2]\}+\frac{1}{2R^2}\sum_{k=-\infty}^{\infty}\operatorname{sinc}^2(\frac{k}{R})[\frac{1}{2}\exp(-\frac{k^2}{b^2})]\delta(f-kf_c)$$

$$\triangleq S_c(f)+\sum_{k=-\infty}^{\infty}S_d(kf_c)\delta(f-kf_c) \cdots\cdots (17)$$

Where $S_c(f)$ and $S_d(f)$ denote the continuous and the discrete spectral components.

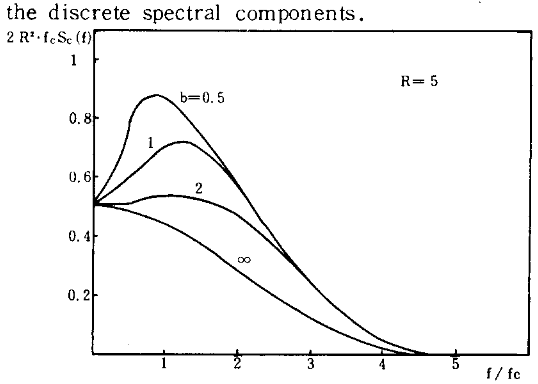

Fig. 1. Normalized continuous spectral density for a random unipolar pulse train with phase jitter.

Fig. 1 shows that the continuous spectral density $S_c(f)$ increases rapidly around $f=f_c$ as the jitter variance increases. The discrete spectral density $S_d(f)$ decreases as the jitter increases as shown in Fig. 2 for $f=f_c$.

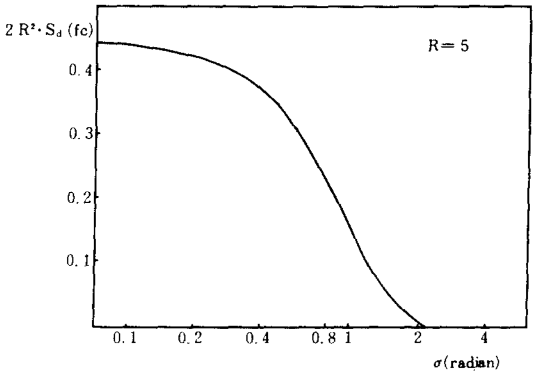

Fig. 2. Normalized discrete spectral density at $f=f_c$ for a random unipolar pulse train with phase jitter.

4. Conclusion

The power spectrum of the jittered random pulse train signal is derived as an extention of Bennett's result. For the unipolar pulse train, it has been shown that as the jitter increases, the continuous power component increases while the discrete power component decreases.

The effect of intersymbol interference was not discussed in the analysis, but the derived spectrum and the figures are expected to be helpful in the design of baseband synchronization systems.

Reference

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