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Jackknife Estimator of Logistic Transformation from Truncated Data

Lee, Won Hyung*

ABSTRACT

In medical follow-up, equipment lifetesting, various military situations, and other fields, one often desires to calculate survival probability as a function of time, p(t). If the observer is able to record the time of occurrence of the event of interest (called a "death"), then an empirical, non-parametric estimate may simply by obtained from the fraction of survivors after various elapsed times. The estimation is more complicated when the data are truncated, i.e., when the observer loses track of some individuals before death occurs. The product-limit method of Kaplan and Meier is one way of estimating p(t) when the mechanism causing truncation is independent of the mechanism causing death.

This paper proposes jackknife estimators of logistic transformation and compares it to the product-limit method.

*) Office of OR/SA, ROKA HQ

A computer simulation is used to generate the times of death and truncation from a variety of assumed distributions.

INTRODUCTION

In medical follow-up, equipment lifetime testing, various military situations, and other fields, it is often desired to estimate the probability of survival as a function of time, p(t), from empirical data. In many situations, the analyst has no prior knowledge of the functional form of p(t), and a non-parametric estimator is required.

In the medical field, one might wish to estimate the probability that a patient survice 1, 2, 3, ... years after a certain surgical procedure for cancer. In electronics, one wishes to estimate the probability of continuous failure-free operation of an equipment for various time durations. In the military, one might be interested in the probability of conducting a certain mission, under specified environmental conditions, without detection by the enemy. The event of interest may be a human death, and equipment malfunction, or a sonar detection. However, following Kaplan and Meier, reference (1), this paper will refer to the event of interest as a "death." The test elements in the sample population may be a human, a radio, or a submarine. This paper will refer to the test element as an "individual." The observations of the data are "complete" if the observer is able to record the time of death for every individual in the sample. The observation may be "truncated" if the observer loses track of some individuals at known ages before death occurs. In the medical example, a surviving patient might move away from the area. In the electronics example, the radio might be destroyed in an airplane crash before any of the components had malfunctioned. In the military example, the exercise might terminate at a pre-planned hour before the submarine had been detected.

If the observations are complete, then the estimation of p(t) is straight-forward. With complete observations, the most obvious estimator of p(t) is simply the fraction of individuals in the sample who had not died by time t. However, there are other estimators for use with complete data. When the observations are incomplete, it is necessary to consider whether the mechanism causing death and truncation are independent. In the electronics example, the mechanisms would be independent if the aircraft crash was definitely not due to radio failure. The mechanisms would be correlated (not independent) if the cause of crash were unknown, but possibly due to radio failure.

If the mechanisms are not independent, then the construction of an appropriate estimator can be difficult. This paper is confined to estimators based on data with independent mechanisms for deaths and truncation. The product-limit estimator of Kaplan and Meier, reference (1), is an accepted method of dealing with the problem of truncated data.

THEORY

To consider survivability as a function of time it is convenient to define a hazard function, h(t). For a test element surviving at time t, h(t) gives the probability of failure per unit time. Thus, the cumulative survival probability may be found by solving

$$Ps(t+dt) = Ps(t) \cdot [1-h(t)dt] \qquad (1)$$

Assuming 100 percent reliability of starting elements,

$$Ps(o) = 1, \quad \text{and} \qquad (2)$$

$$Ps(t) = e^{-\int_0^t h(x)\,dx} \qquad (3)$$

gives the general expression for survival probability for a single element. Without assuming a specific analytic form for h(t), it is possible to estimate Ps(t) empirically.

There are two approaches to the problem:

(1) To estimate the probability of survival to an arbitrarily selected set of times, or (2) to estimate the survival probability at the time of observed failures. In either case one must make point-wise estimates of the survival curve. Only if an analytic form of h(t), or Ps(t), is assumed can an estimate of the entire curve be derived.

In the first case, one merely divides the number of entries by the number of survivors at the appropriate time. The second approach is more useful if truncated tests are included in the data. The second approach provides a distribution free estimate of survival probability subject to one restriction on h(t):

$$\int_0^\infty h(t) = \infty \tag{4}$$

This restriction is a mild one for many situations of interest. If N test elements start at time zero, then the probability of all surviving to time t is $e^{-[N\int_0^t h(x)\,dx]}$ The probability of a failure in the interval (t, t+dt) is $Nh(t)dt$ in the limit of small dt. The expected value of Ps at the time of the first failure, t', is thus

$$E(Ps(t')) = \int_0^\infty Ps(t)\cdot e^{-[N\int_0^t h(x)\,dx]} Nh(t)dt$$

The change of variable

$$\alpha = \int_0^t h(x)dx$$

leads to

$$E(Ps(t')) = N\int_0^\infty e^{-(N+1)\alpha}\, d\alpha \tag{5}$$

and

$$E(Ps(t')) = \frac{N}{N+1} \tag{6}$$

Reference (6) state with proof the more general relation

$$E(Ps(t')) = \frac{N-r+1}{N+1} \tag{7}$$

Where t' here is the time of the r^{th} observed failure. Equation (6) suffices if there are no truncated data.

If all aborts and late entries occur at the time of failure (as might be assumed in the case of grouped data), then at the time of n^{th} failure (tn)

$$Ps(tn) = \prod_{i=1}^{n} \left(\frac{Ni}{Ni+1}\right) \qquad (8)$$

is the appropriate estimate, with Ni the number of elements starting the time interval terminated by the i^{th} failure.

The variance associated with the estimate in equation (8) is

$$\text{Var}\,[\,Ps\,(t_n)\,] = \prod_{i=1}^{n}\left(\frac{Ni}{Ni+2}\right) - \prod_{i=1}^{n}\left(\frac{Ni}{Ni+1}\right)^2 \qquad (9)$$

If truncated runs begin or end at times other than when a failure occurs, equations (8) and (9) are not quite correct. If Ps(t) is assumed to follow a simple exponential decay curve with the understanding that Ni is the average number of surviving test elements in the interval between the $(i-1)^{th}$ failure and the i^{th} failure.

THE JACKKNIFE ESTIMATOR

We will assume that we observed, or have generated in a simulation, a survival probability $p(t_j)$, $j = 1, \ldots, n$, from various sample sizes. Furthermore we have some parameter or characteristic $p(t_j)$ of the sample size which we wish to estimate with an estimator $\tilde{p}(t_j)$. The jackknife estimator $\tilde{p}(t,n)$ described below is an approximately unbiased estimator of $p(t_j)$. A modification of it has other useful properties.

$\tilde{P}_{-i}(t,n-1)$ is the estimator from the sample of n of the Xi's with the i^{th} value deleted from the sample.

$$\tilde{P}_i(t,n) = n\tilde{P}(t,n) - (n-1)\tilde{P}_{-1}(t,n-1) \quad i = 1, \cdots, n$$

$$\tilde{P}(t,n) = \frac{1}{n}\sum_{i=1}^{n}\tilde{P}_i(t,n) = n\tilde{P}(t,n) - \frac{n-1}{n}\sum_{i=1}^{n}\tilde{P}_{-1}(t,n-1)$$

the $\tilde{P}_i(t,n)$, called the PSEUDO-Values.

The PSEUDO-Values can be used to obtain variance estimates of $\tilde{P}(t,n)$ and to set approximate confidence limits, using Student's t. The idea is that the PSEUDO-Values will be approximately independently and normally distributed. The jackknife estimator $\tilde{P}(t,n)$ is a sample average so we form an estimate $S^2_{\tilde{P}(t,n)}$ of its variance given by the following relationship (Miller, 1974):

$$S^2 = \frac{\sum \tilde{P}_i^2(t,n) - \frac{1}{n}(\sum \tilde{P}_i(t,n))^2}{n-1}$$

$$S^2_{\tilde{P}(t,n)} = \frac{S^2}{n}$$

This procedure is particularly useful if the number of data points is small, but it must be used with care. Note, that the estimator $\tilde{P}(t,n)$ is designed to eliminate a $\frac{1}{n}$ bias term in the estimator $\tilde{P}(t,n)$. Of course the computational aspects of the complete jackknife can be quite onerous, especially if $\tilde{P}(n)$ were, say, a complicated maximum likelihood estimator. Miller, reference (4) has shown the product limit estimator is its own jackknife.

LOGISTIC TRANSFORMATION

Although one can legitimately jackknife the Kaplan-Meier estimate directly, there is some reason to believe that a preliminary transformation will give improved results. Consequently, consider the transformation

$$\ell = \ln \left(\frac{\widetilde{P}(t)}{1 - \widetilde{P}(t)} \right)$$

and notice that where the range of $\widetilde{P}(t)$ is from zero to unity, the above transformation makes the range of ℓ run from $-\infty$ to ∞. The procedure utilized will be as follows.

(A) Compute the overall estimate at a time point t, using all N data points, and using a "continuity" correction that has the effect of removing the effect of a zero in the logarithm (see D.R. Cox, Analysis of Binary Data, Methuen Monograph):

$$\ell_N = \ln \left(\frac{\widetilde{P}_N(t) + \frac{1}{2N}}{1 - \widetilde{P}_N(t) + \frac{1}{2N}} \right)$$

(B) Compute the ℓ-values by leaving out each data point in turn when computing P(t):

for i = 1, 2, ..., N.

$$\ell_{N-1,i} = \ell n \left(\frac{\widetilde{P}_{N-1,-i}(t) + \frac{1}{2(N-1)}}{1 - \widetilde{P}_{N-1,i}(t) + \frac{1}{2(N-1)}} \right)$$

(C) Form the PSEUDO-Values

$$Z_i = N\ell_N - (N-1)\ell_{N-1,-i}$$

(D) Compute $\overline{Z}$, S_z^2

(E) Put approximate confidence $(1-\alpha)$. 100% limits on $E[\ell]$ as follows $L \leqq E[\ell] \leqq H$

where $H(L) = Z + (-) t_{1-\alpha} (N-1) \sqrt{\frac{S_Z^2}{N}}$

(F) Transform bash to obtain

$$\frac{e^L}{1+e^L}, \quad \text{and} \quad \frac{e^H}{1+e^H}$$

The true value, $P(t)$, should be enclosed between these levels for roughly $(1-\alpha)$. 100% of all samples. The coverage properties of this procedure will now be checked by simulation: Successive sample of size N will be selected, the jackknife limits H and L will be computed for each, and a check will be made as to whether

$$\frac{e^L}{1+e^L} \leqq P(t) \leqq \frac{e^H}{1+e^H} \text{ or not.}$$

COMPARISON OF THE PRODUCT-LIMIT ESTIMATOR AND JACKKNIFE ESTIMATOR OF LOGISTIC TRANSFORMATION

A hypotetical data base, consisting of five individuals, is used to illustrate each of the estimators. This sample data base is as follows:

Individual	Time of Death	Time of Truncation
A	1	-
B	Unknown ($>$ 2)	2
C	3	-
D	Unknown ($>$ 6)	6
E	7	-

The data have been arranged in time sequence of the death and truncation events. In the medical example, the data might indicate that patients A, C and E were observed to die exactly 1, 3 and 7 years, respectively, after their surgery. However, B and D moved away or otherwise became unavilable to the observer at these times. Further, the cause of the unobservability is unrelated to the patient's health and life expectancy.

1. The Product-limit estimator,"$\widetilde{P}_1(t)$"

$\widetilde{P}1(t)$ is the product-limit estimate. Kaplan and Meier, reference (1), have shown that this is the maximum likelihood estimator. The observed events, both deaths and truncations, are arranged in increasing order of occurrence: t_1, t_2, ..., t_N,; where N is the number of individuals in the sample.

Let p(ti) denote the cumulative probability of survival of an individual from time zero to time ti. Let p(t/ti) denote the conditional probability of surviving to time t (>ti), given that the individual has already survived to time ti. Then,

$$\tilde{P}1(ti) = P1(ti-1) \cdot P1(ti/ti) \qquad (E-1)$$

If we define to = 0 and p(o) = 1, then

$$\tilde{P}1(ti) = \prod_{j=1}^{i} P1(t_j / t_{j-1}) \qquad (E-2)$$

The product limit estimator is in the form of equation (E-2) with

$$\tilde{P}_1(t_j \mid t_{j-1}) = \begin{cases} \dfrac{Nj}{Nj} = 1 & \text{if the event at } t_j \text{ is truncation} \\[2ex] \dfrac{Nj-1}{Nj} & \text{if the event at } t_j \text{ is a death} \end{cases} \qquad (E-3)$$

Here N_j is the number of individuals observed surviving in the interval $t_{j-1} < t < t_j$. This formulation causes the product limit estimator to be insensitive to the exact time of the censoring events. The estimator is unity from time zero to the time of the first event, t_1, reflecting the fact that all individuals in our example are observed to live until at least time t_1.

- If the event at time t_1 is a truncation, then the estimator remains at unity at least time t_2. Again, no deaths are observed in the sample before t_2.
- If the event at time t_1 is a death, then the estimator drops to $(N-1)/N$. This drop reflects the observed death of 1/N of the survival sample just prior to t_1.

Values of the estimator $\tilde{P}1$ are calculated iteratively at successive values of ti(i=1,2,...,N).

The size of the survival sample declines as truncations and deaths remove individuals from observation. For the hypothetical data base listed above, one obtains:

t	$\tilde{P}1(t)$
0 - 1	5/5 = 1.0
1 - 2	4/5 = 0.8
2 - 3	(4/5) x (3/3) = 0.8
3 - 6	(4/5) x (2/3) = 0.533
6 - 7	(8/15) x (1/1) = 0.533
7 - 00	(8/15) x (0/1) = 0.0

If the last event in the sample is a truncation rather than a death, then the modified data give the following estimate, i.e., individual E had disappeared from the observer at time 6.5 (so that the fact of E's death at time 7 is unknown).

t	$\tilde{P}1(t)$ - Modified Data
0 - 1	1.0
1 - 3	0.8
3 - 6.5	0.533

Since the time of the death for individual E is now unknown, one can only estimate that:

$$0 \leq \tilde{P}1(t) \leq 0.533 \quad \text{for} \quad t > 6.5$$

2. The logistic transformation estimator

$\ell_{N-1,-i}$ is the logistic transformation estimator from the sample N of the Xi's with the i^{th} value deleted from the sample.

$$\ell_{N-1,-i} = \ell n \left(\frac{\tilde{P}_{N-1,-i}(t) + \frac{1}{2(N-1)}}{1 - \tilde{P}_{N-1,-i}(t) + \frac{1}{2(N-1)}} \right)$$

$\ell_{N-1,-i}$;

t	1	2	3	4	5
t1	3.04	0.98	0.98	0.98	0.98
t2	3.04	0.98	0.98	0.98	0.98
t3	0.63	0	0.98	-0.46	-0.46
t4	0.63	0	0.98	-0.46	-0.46
t5	-3.04	-3.04	-3.04	-3.04	-1.89

$$Z_i = N\ell_N - (N-1)\,\ell_{N-1,-i}$$

$$= N\ell n \left(\frac{\tilde{P}_N(t) + \frac{1}{2N}}{1 - \tilde{P}_N(t) + \frac{1}{2N}} \right) - (N-1)\,\ell n \left(\frac{\tilde{P}_{N-1,-i}(t) + \frac{1}{2(N-1)}}{1 - \tilde{P}_{N-1,-i}(t) + \frac{1}{2(N-1)}} \right)$$

Z_i (N) are called PSEUDO-Values of logistic transformation, the following values are calculated:

Z_i :

t	1	2	3	4	5
t1	-0.65	2.198	2.198	2.198	2.198
t2	-6.05	2.198	2.198	2.198	2.198
t3	-1.9	0.606	-3.314	2.446	2.446
t4	-1.9	0.606	-3.314	2.446	2.446
t5	-3.0626	-3.0626	-3.0626	-3.0626	-7.162

Average of the PSEUDO-Values

$$\bar{Z} = \frac{1}{N} \sum_{i=1}^{N} Z_i$$

Invert to find jackknife estimator of logistic transformation

$$\bar{Z} = \ell n \left(\frac{\tilde{P}(t) + \frac{1}{2N}}{1 - \tilde{P}(t) + \frac{1}{2N}} \right)$$

$$\tilde{P}(t) = \frac{(1 + \frac{1}{2N})\, e^{\bar{Z}} - \frac{1}{2N}}{1 + e^{\bar{Z}}}$$

called the jackknife estimator of logistic transformation

Variance of the Zi

$$S_Z^2 = \mathrm{Var}(Z) = \frac{1}{n-1} \sum_{i=1}^{n} Z_i - \bar{Z}$$

The following values are calculated:

t	$\bar{Z}$	$\tilde{P}(t)$	Var
t1	0.5484	0.646	13.6
t2	0.5484	0.646	13.6
t3	0.568	0.516	6.727
t4	0.0568	0.516	6.727
t5	-3.882	0	3.361

Confidence Interval

The jackknife estimator for estimating variability and giving confidence interval.

Tukey, reference (3) has suggested that in the jackknife procedure, we consider the PSEUDO-Values Zi(N) as approximately independent and identically distributed and consequently, since $\bar{Z}$ is an average of the Zi(N), proceed as if

$$\frac{N^{\frac{1}{2}}\bar{Z} - \ell_N}{\left\{\frac{1}{N-1}\sum_{i=1}^{N}(Z - \bar{Z})^2\right\}^{\frac{1}{2}}}$$

has t-distribution with N-1 d.F.

If the Zi are approximately normal variates (Miller has shown) confidence bands for the unknown $\widetilde{P}(t)$ are given, as for the mean of any normal variate when estimated from sample size N.

$$\bar{Z} \pm \frac{S_Z}{\sqrt{N}} t_{1-\alpha/2}(N-1) \qquad (D-1)$$

i.e.

$$\bar{Z} - \frac{S_Z}{\sqrt{N}} t_{1-\alpha/2}(N-1) \leqq \ell n\left(\frac{\widetilde{P}(t) + \frac{1}{2N}}{1 - \widetilde{P}(t) + \frac{1}{2N}}\right) \leqq \bar{Z} + \frac{S_Z}{\sqrt{N}} t_{1-\alpha/2}(N-1)$$

$$\underline{L}(n) = \bar{Z} - \frac{S_Z}{\sqrt{N}} t_{1-\alpha/2}, \quad \bar{L}(n) = \bar{Z} + \frac{S_Z}{\sqrt{N}} t_{1-\alpha/2}$$

$$\frac{\left(1 + \frac{1}{2N}\right) e^{\underline{L}(N)} - \frac{1}{2N}}{1 + e^{\underline{L}(N)}} \leqq \widetilde{P}(t) \leqq \frac{\left(1 + \frac{1}{2N}\right) e^{\bar{L}(N)} - \frac{1}{2N}}{1 + e^{\bar{L}(N)}}$$

Results of the Simulation

The jackknife procedure may be validated, in an empirical sense, by sampling experiments or computer simulation in the following manner. First, times of censoring and death are obtained by drawing random numbers from postulated distributions. Second, the jackknifed estimator of the logistic-transformed product-limit estimation is found and confidence limits are computed by the method of Tukey, reference (3). Since the true value of survival function, P(t), is known, so is the theoretical value of A. The jackknife confidence intervals can be checked for coverage: if $L_\alpha \leqq A \leqq H_\alpha$ then the particular interval covers, while otherwise (if $A < L_\alpha$ or $H_\alpha < A$) it does not cover. Finally, the above procedure can be repeated many times (say 1000) and the fraction of repetitions which contains the true value of A is recorded.

This fraction of the coverage should desirably be close to (1- α), the nominal confidence level. The jackknife confidence limit procedure can be said to be robust of validity, ref (5), if the actual coverage is close to the nominal coverage, 1- α , for a various distributions. Such seems to be true for large N ($N \geqq 50$). However, the jackknife confidence limits do not cover accurately when the true value of P(t) is close to unity.

The following tables illustrate confidence limits of jackknife method of product limit ($\widetilde{P1}(t)$).

Table 1

Simulation Experiments Validating Table
95% Confidence Limits (t value=2.776)

n=5. Dist.of Death	ensoring Dist.of	trunc.parameter distinguish	True Value	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
Expo-nential	Expo-nential	2.0	Upper Int.	0.962	0.935	0.917	0.912	0.922	0.945	0.971	0.995	1.0
			Lower Int.	0.709	0.515	0.366	0.259	0.173	0.115	0.068	0.034	0.009
			Converage	56.735	66.122	75.408	85.714	90.816	96.939	98.980	95.918	96.939
		1.3333	Upper Int.	0.962	0.935	0.918	0.914	0.923	0.942	0.969	0.992	1.0
			Lower Int.	0.714	0.559	0.422	0.313	0.238	0.166	0.10	0.059	0.068
			Converage	57.209	65.116	72.093	79.070	86.047	93.029	95.023	05.023	86.047
Uni-form	Expo-nential	4.0	Upper Int.	0.955	0.910	0.871	0.846	0.825	0.819	0.827	0.837	0.855
			Lower Int.	0.744	0.552	0.407	0.301	0.217	0.163	0.113	0.078	0.047
			Converage	50.792	68.358	74.194	85.630	92.669	96.774	92.150	95.308	88.886
		0.6667	Upper Int.	0.957	0.923	0.901	0.893	0.898	0.904	0.911	0.915	0.932
			Lower Int.	0.775	0.618	0.494	0.393	0.324	0.276	0.202	0.166	0.025
			Converage	54.545	65.545	73.636	82.727	92.273	94.091	86.364	87.27	72.787
Expo-nential	Uni-form	2.0	Upper Int.	0.964	0.931	0.911	0.901	0.906	0.917	0.940	0.963	**
			Lower Int.	0.760	0.558	0.419	0.305	0.231	0.171	0.125	0.122	**
			Converage	55.887	56.259	72.340	82.496	87.943	91.489	85.668	87.943	**
		1.3333	Upper Int.	0.957	0.924	0.902	0.891	0.897	0.914	0.979	**	**
			Lower Int.	0.757	0.580	0.456	0.380	0.305	0.256	0.247	**	**
			Converage	45.887	54.255	72.340	85.496	88.15	93.52	96.688	**	**
Uni-form	Uni-form	4.0	Upper Int.	0.948	0.906	0.862	0.836	0.821	0.816	0.821	0.837	0.863
			Lower Int.	0.718	0.533	0.385	0.291	0.216	0.155	0.112	0.080	0.049
			Converage	53.117	61.039	68.571	87.987	93.506	96.429	91.558	95.130	89.844
		1.0	Upper Int.	0.953	0.917	0.893	0.882	0.883	0.891	0.907	0.924	0.938
			Lower Int.	0.751	0.60	0.496	0.399	0.318	0.276	0.251	0.131	0.049
			Converage	58.0	70.0	78.333	78.667	78.667	80.0	70.0	73.333	66.667

Table 3

Simulation Experiments Validating Table
95% Confidence Limits (t value =2.093)

n=25

Dist. of Die	Dist. of Trunc.	Trunc. parameter	distinguish \ True Value	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
Expo-nential	Expo-nential	2.0	Upper Int.	0.986	0.948	0.893	0.833	0.774	0.717	0.682	0.798	0.805
			Lower Int.	0.787	0.603	0.477	0.372	0.283	0.199	0.126	0.060	0.0
			Converage	68.148	93.827	97.778	98.765	98.672	98.272	99.259	98.025	95.309
		1.333	Upper Int.	0.941	0.897	0.848	0.787	0.746	0.702	0.693	0.731	0.826
			Lower Int.	0.778	0.612	0.501	-.411	0.325	0.245	0.173	0.112	0.058
			Converage	66.260	91.057	97.154	98.374	98.780	95,935	96.341	93.496	80.488
		0.667	Upper Int.	0.967	0.910	0.851	0.794	0.753	0.745	0.785	0.825	0.892
			Lower Int.	0.850	0.696	0.582	0.478	0.384	0.302	0.211	0.159	0.103
			Converage	67.742	90.323	90.323	93.548	93.548	90.323	87.097	74.194	58.065
Expo-nential	Uni-form	4.0	Upper Int.	0.948	0.911	0.864	0.809	0.748	0.689	0.636	0.614	0.683
			Lower Int.	0.751	0.578	0.469	0.376	0.293	0.218	0.149	0.091	0.045
			Converage	66,821	93.968	97.681	99.768	98.840	99.536	98.840	98.376	93.039
		1.333	Upper Int.	0.922	0.860	0.792	0.730	0.675	0.643	0.725	**	**
			Lower Int.	0.836	0.708	0.601	0.513	0.431	0.369	0.327	**	**
			Converage	66.260	91.057	97.154	98.374	98.780	95.935	96.341	**	**
Uni-form	Expo-nential	2.00	Upper Int.	0.951	0.878	0.799	0.714	0.627	0.541	0.455	0.372	0.309
			Lower Int.	0.393	0.299	0.274	0.250	0.221	0.186	0.146	0.100	0.054
			Converage	85.985	87.121	90.278	91.919	91.919	93.813	95.581	97.854	88.005
		1.00	Upper Int.	0.950	0.877	0.796	0.717	0.638	0.559	0.490	0.446	0.428
			Lower Int.	0.402	0.306	0.275	0.249	0.218	0.181	0.140	0.096	0.056
			Converage	83.770	86.721	89.344	90.358	93.934	95.738	95.525	98.689	86.866
Uni-form	Uni-form	2.00	Upper Int.	0.952	0.879	0.797	0.761	0.624	0.541	0.460	0.394	0.347
			Lower Int.	0.406	0.303	0.174	0.250	0.220	0.186	0.146	0.100	0.057
			Converage	84.253	89.367	89.906	90.675	92.059	94.612	95.577	97.443	86.945

Table 2

Simulation Experiments Validating Table
95% Confidence Limits (t value=2.262)

n=10 Dist. of Death	Dist. of Censoring	trunc. parameter	distinguish / True Value	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
Expo-nential	Expo-nential	4.0	Upper Int.	0.975	0.953	0.922	0.887	0.859	0.844	0.854	0.896	0.966
			Lower Int.	0.840	0.640	0.455	0.336	0.242	0.163	0.092	0.033	0.0
			Converage	52.667	70.933	78.933	91.733	95.733	97.60	97.067	97.067	96.8
		1.0	Upper Int.	0.966	0.930	0.891	0.861	0.851	0.858	0.892	0.938	0.976
			Lower Int.	0.582	0.359	0.241	0.176	0.125	0.984	0.058	0.032	0.015
			Converage	62.0	87.50	94.643	98.214	100.0	100.0	100.0	98.214	98.214
Uni-form	Expo-nential	1.0	Upper Int.	0.954	0.899	0.843	0.792	0.753	0.730	0.731	0.745	0.760
			Lower Int.	0.612	0.410	0.305	0.250	0.204	0.160	0.120	0.083	0.060
			Converage	57.798	77.982	89.297	96.330	99.083	98.165	99.083	96.024	76.453
		2.0	Upper Int.	0.956	0.897	0.834	0.772	0.716	0.673	0.646	0.637	0.649
			Lower Int.	0.615	0.391	0.286	0.227	0.138	0.151	0.120	0.089	0.058
			Converage	53.542	78.125	84.167	94.167	98.333	98.542	99.167	97.917	82.083
Expo-nential	Uni form	4.0	Upper Int.	0.975	0.948	0.913	0.874	0.840	0.821	0.828	0.865	0.937
			Lower Int.	0.546	0.295	0.165	0.100	0.068	0.048	0.030	0.012	0.003
			Converage	60.0	86.154	96.923	98.174	100.0	100.0	100.0	100.0	100.0
		2.0	Upper Int.	0.940	0.896	0.855	0.814	0.787	0.785	0.806	0.851	**
			Lower Int.	0.860	0.709	0.578	0.468	0.379	0.308	0.251	0.168	**
			Converage	57.6	51.2	78.933	91.733	96.0	98.133	97.333	96.8	**
Uni-form	Uni-form	1.0	Upper Int.	0.949	0.896	0.837	0.794	0.767	0.761	0.779	0.800	0.840
			Lower Int.	0.596	0.423	0.324	0.265	0.225	0.182	0.150	0.130	0.097
			Converage	59.091	79.545	93.182	93.182	100.0	97.727	97.727	72.727	56.818
		4.0	Upper Int.	0.957	0.902	0.836	0.767	0.703	0.647	0.603	0.583	0.589
			Lower Int.	0.602	0.383	0.272	0.220	0.184	0.153	0.121	0.089	0.055
			Converage	57.093	79.159	87.566	94.921	99.124	100.0	100.0	98.949	86.165

Table 4

Simulation Experiments Validating Table
95% Confidence Limits (ι value =2.010 1)

n=50

Dist. of Death	Dist. of Censoring	Trunc. parameter / distinguish	True Value	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
Expo-nential	Uniform	1.0	Upper Int.	0.938	0.856	0.772	0.692	0.626	0.636	**	**	**
			Lower Int.	0.300	0.302	0.290	0.268	0.238	0.215	**	**	**
			Converage	94.650	94.870	95.140	95.520	95.910	96.720	**	**	**
		1.5	Upper Int.	0.950	0.876	0.799	0.717	0.636	0.566	0.540	**	**
			Lower Int.	0.238	0.245	0.244	0.228	0.202	0.168	0.132	**	**
			Converage	94.860	95.120	95.210	95.830	95.960	96.120	96.120	**	**
		3.0	Upper Int.	0.968	0.908	0.839	0.766	0.689	0.612	0.541	0.503	0.625
			Lower Int.	0.121	0.142	0.153	0.149	0.134	0.111	0.042	0.049	0.020
			Converage	94.275	95.237	95.237	95.618	95.820	95.820	96.012	96.947	95.846
Expo-nential	Expo-nential	0.6667	Upper Int.	0.959	0.892	0.822	0.753	0.689	0.648	0.670	0.756	0.864
			Lower Int.	0.201	0.199	0.198	0.182	0.155	0.121	0.083	0.046	0.021
			Converage	92.381	94.325	94.875	95.312	95.312	96.012	96.463	96.463	95.212
		1.3333	Upper Int.	0.970	0.913	0.848	0.780	0.711	0.646	0.602	0.635	0.798
			Lower Int.	0.128	0.136	0.144	0.137	0.121	0.097	0.066	0.034	0.009
			Converage	94.568	94.761	95.327	95.327	96.032	96.032	95.882	95.882	94.895
		2.0	Upper Int.	0.974	0.920	0.858	0.791	0.721	0.653	0.594	0.585	0.734
			Lower Int.	0.103	0.114	0.124	0.120	0.107	0.087	0.061	0.030	0.007
			Converage	94.523	94.876	95.038	95.523	95.823	96.720	96.720	95.823	94.028
Uni-form	Uni-form	1.0	Upper Int.	0.937	0.855	0.769	0.681	0.598	0.520	0.465	0.472	0.599
			Lower Int.	0.318	0.312	0.301	0.277	0.246	0.209	0.165	0.122	0.076
			Converage	94.667	95.222	95.667	95.778	95.778	96.033	96.112	94.778	94.556
		4.0	Upper Int.	0.936	0.853	0.766	0.677	0.585	0.490	0.393	0.295	0.203
			Lower Int.	0.296	0.303	0.295	0.275	0.247	0.211	0.168	0.118	0.058
			Converage	94.826	95.362	96.012	96.723	96.723	97.023	97.023	96.052	94.234

REFERENCES

1. "Nonparametric Estimation from Incomplete Observation," E. L. Kaplan and Paul Meier, Meier, AMERICAN STATISTICAL ASSOCIATION JOURNAL June 1958.

2. "Electronic Reliability," G. R. Herd, ARINC MONOGRAPH NO. 3, 1 May 1956.

3. BIOMETRICS, J. W. Tukey, 1951.

4. "Jackknife Censored Data," Rupert G. Miller, Jr., Technical Report, Stanford University, 1976.

5. "Modeling and Estimating System Availability," Donald P. Gaver, NPS 55-77-4, January 1977.

6. "Nonparametric Estimation from Censored Data," Lee Won Hyung, THESIS, Naval Postgraduate School, March 1978.