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Stochastic Duels with Multiple Hits, and Fixed Duel Time

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ABSTRACT

A stochastic fundamental duel with continous interfiring times is considered for including the kill effect of multiple hits and fixed duel time. Two alternatives, "vital hit" and "damage coefficient" approaches, are developed. Since a large quantity of ammunition is consumed when a sure kill is obtained through repetitive multiple hits, limitation of initial ammunition supply is included in the stochastic duel models with multiple hits and fixed duel time.

General solutions are obtained and examples with negative exponential interfiring times and geometric ammunition supply are given.

1. INTROUDCTION

In the fundamental duel [9], the assumption that the combatant who scores a hit first is the winner implies that a hit always results in a kill. However, operational experience indicates that this may not be true.

To remove the assumption that a sure kill is achieved by a single hit,

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Bhasyam 〔3〕 first constructed stochastic duel models with "lethal dose" using a difference-differential equation technique. However, his approach is limited since it is difficult to include additional factors in the duel model because each additional factor must be used as a state variable in the difference-differential equations.

In Section Ⅱ, two alternatives are developed, "vital hit shot" and "damage coefficient" approaches which include the kill effects of multiple hits in the fundamental duel for the case where ammunition supply is unlimited and only a single shot can be fired each time 〔7〕.

In the "vital hit" approach it is assumed that a sure kill is obtained only at the vital hit shot and that the number of shots required to get a sure kill follows a negative binomial distribution. In the "damage coefficient" approach, on the other hand, it is assumed that regardless of the number of previous hits the chance of destroying the target with any hit shot is always the same. In this approach renewal theory is utilized. Both approaches lead to the same result if random vital hit shot follows a geometric distribution.

One important aspect to be considered in stochastic duels is the limitation on ammunition supply, especially for the case where a sure kill is obtained through multiple hits and therefore a large quantity of ammunition is consumed. Thus, in Section Ⅲ, the kill effect of multiple hits is incorporated into the fundamental duel for the case where ammunition supply is limited and only single firing is allowed.

II. Multiple Hits with Unlimited Ammunition Supply

II. 1. Vital Hit Approach[7]

It is assumed that (1) both A and B start the duel with unloaded weapons and unlimited time, and have fixed single shot hit probabilities p_A and p_B, and interfiring time probability density functions (pdf) $f_A(t)$ and $f_B(t)$ respectively, and (3) A kills B exactly at A's r-th hit (A's vital hit) on B and A is killed exactly at B's k-th hit.

We here define that

$F_A(t)$: A's diststribution function of interfiring times on passive target B.

$H_{A_r}(t)$: the distribution function that A scores the vital hit first in $(0, t]$ with $(r-1)$ hits and x misses prior to the vital hit.

$h_{A_r}(t)$: the pdf that A scores the vital hit first in $(t, t+dt]$.

$a^{(x)}(t)$: the x-fold convolution of function $a(t)$.

$a^*(s)$: the Laplace transform of function $a(t)$.

Then, we can express the probability $dH_{A_r}(t)$ by techniques similar to williams and Ancker [9] as

$$(1)\quad dH_{A_r}(t) = \sum_{x=0}^{\infty} \binom{x+r-1}{x} \cdot p_A^r \, q_A^x \, dF_A^{(x+r)}(t)$$

If we assume that $F_A(t)$ is absolutely continuous and

$$dF_A^{(x)}(t) = f_A^{(x)}(t)\,dt,$$

we obtain Laplace transform of equation (1) as

$$(2)\quad h_{A_r}^*(s) = \sum_{x=0}^{\infty} \binom{x+r-1}{x} \; p_A^r \, q_A^x \, (f_A^*(s))^{x+r} = \left(\frac{p_A f_A^*(s)}{1-q_A f_A^*(s)}\right)^r$$

$h_{B_k}^*(s)$ is similarly defined for duelist B.

Equation (2) coincides with Bhashyam's result [3].
However, our approach is simple and concise, whereas Bhashyam's derivation involves complicated difference-differential equations.

A general solution for stochastic duels consists of finding the winning probability of a given side and the probability of a draw. For duelist A, if t is time until A kills a passive target, the probability $P(\tau; A)$ that A kills B before he is killed when the duel time is limited to τ can be given by

(3) $$P(\tau\ ;\ A) = \int_0^T h_{A_r}(t)\left[\int_t^\infty h_{B_k}(\tau)\,d\tau\right]dt = \int_0^T k(t;\overline{A}_r, B_k)\,dt$$

where $k(t:\overline{A}_r, B_k)\,dt$ is the probability that A scores r-th hit before B obtains k-th hit in time interval $(t, t+dt]$ and the bar (–) indicates the duelist who has the vital shot first.

Then, we obtain $P^*(s;A)$ by techniques similar to Kwon and Bai [5, 6, 7] and Bai and Kwon [2] as

(4) $$P^*(s\ ;\ A) = (\tfrac{1}{s})\ k^*(s;\overline{A}_r, B_k) = (\tfrac{1}{s})\left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h^*_{A_r}(s+z)\ h^*_{B_k}(-z)\frac{dz}{z}\right]$$

The probability $P(\tau\ ;\ AB)$ of a draw is

(5) $$\begin{aligned}P(\tau\ ;\ AB) &= \int_T^\infty\int_T^\infty h_{A_r}(t)\ h_{B_k}(\tau)\,dt\ d\tau \\ &= \int_T^\infty\int_t^\infty h_{A_r}(t)\ h_{B_k}(\tau)\,d\tau\ dt + \int_T^\infty\int_\tau^\infty h_{A_r}(t)\ h_{B_k}(\tau)\,dt\ d\tau \\ &= \int_T^\infty \{k(t\ ;\ \overline{A}_r, B_k) + k(t\ ;\ A_r, \overline{B}_k)\}\,dt\ .\end{aligned}$$

From the final value theorem [4], *i.e.*,

(6) $$\lim_{s\to 0} s\cdot P^*(s\ ;\ A) = \lim_{T\to\infty} P(\tau, A)$$

and equation (4), the probability $P(A)$ that A wins when the duel time is unlimited can be written as

(7) $$\begin{aligned}P(A) &= \lim_{T\to\infty} P(\tau\ ;\ A) \\ &= \int_0^\infty h_{A_r}(t)\left[\int_t^\infty h_{B_k}(\tau)\,d\tau\right]dt \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h^*_{A_r}(z)\ h^*_{B_k}(-z)\ \frac{dz}{z}\end{aligned}$$

For B, $P(r;B)$, $P^*(s;B)$ and $P(B)$ can be similarly obtained.

If the number R of hits required to obtain the vital hit is assumed to be random and geometrically distributed with parameter $c_A = Pr$ (kill/hit), that is, $Pr(R=r) = c_A(1-c_A)^{r-1}$, the probability $k_{A_r}(t)\,dt$ that A will kill B exactly at the r-th hit in time interval $(t, t+dt]$ can be expressed as

$$(8)\quad k_{A_r}(t)\,dt = c_A(1-c_A)^{r-1}\,h_{A_r}(t)\,dt.$$

We also have

$$(9)\quad k^*_{A_r}(s) = \left(\frac{c_A}{1-c_A}\right)\left(\frac{p_A(1-c_A)\,f_A^*(s)}{1-q_A f_A^*(s)}\right)^r.$$

By summing equation (9) over r, the Laplace transform $k_A^*(s)$ of kill pdf $k_A(t)$, which is a weighted average of $k_{A_r}(t)$, can be obtained as

$$(10)\quad k_A^*(s) = \left(\frac{c_A}{1-c_A}\right)\left[\frac{p_A(1-c_A)\,f_A^*(s)}{1-q_A f_A^*(s)}\right]\left[\frac{1}{1-\dfrac{p_A(1-c_A)\,f_A^*(s)}{1-q_A f_A^*(s)}}\right]$$

$$= \frac{p_A c_A f_A^*(s)}{1-(1-p_A c_A)\,f_A^*(s)}$$

similarly for B,

$$(11)\quad k_B^*(s) = \frac{p_B\, c_B\, f_B^*(s)}{1-(1-p_B c_B)\,f_B^*(s)}$$

With arguments similar to the ones leading to equations (4) and (7), we have

$$(12)\quad P^*(s;A) = \left(\frac{1}{s}\right) k^*(s;\bar{A},B)$$

$$= \left(\frac{1}{s}\right)\left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k_A^*(s+z)\;k_B^*(-z)\,\frac{dz}{z}\right]$$

and

$$(13)\quad P(A) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k_A^*(z)\;k_B^*(-z)\,\frac{dz}{z}.$$

II.2. Damage Coefficient Approach

Equation (10) can also be obtained by the damage coefficient approach. Morse and Kimball [8] expressed the probability $P(A)$ that A destroys his opponent as

$$(14)\quad P(A) = \sum_{i=1}^{\infty} P_{A_i} \cdot c_{A_i}$$

where P_{A_i} is the probability of i hits by a given firing method of duelist A and c_{A_i}, called a damage coefficient, is the conditional kill probability given i hits. As a matter of operational experience it has been found that damage coefficient in many cases can be given by the law of composition of independent probabilities as

$$(15)\quad c_{A_i} = 1-(1-c_{A_1})^i,\ i=1,2,3,\ \ldots\ldots$$

which is based on "vital spot" hypothesis; that is, a target will only be destroyed if it is hit on its critical spots and hits other than these spots will damage, but not destroy the opponent.

With the kill effects of multiple hits expressed as in equations (14) and (15) and using renewal theory, equation (10) can be derived as follows;

$N_A(\tau)$: the number of shots fired by contestant A during time interval $(0,\tau]$.

$T_A(n)$: total elasped time in delivering n-shots to the opponent.

$H_A(\tau)$: the number of successful hit shots in $(0,\tau]$.

Then, we have

$$(15)\quad P[N_A(\tau)=n] = P[N_A(\tau)<n+1] - P[N_A(\tau)<n]$$
$$= F_n(\tau) - F_{n+1}(\tau).$$

where

$$F_n(\tau) = P[T_A(n) \leq \tau] = \int_0^T f_A^{(n)}(t)\,dt,$$

and

$$(16)\quad P[H_A(\tau)=i] = \sum_{n=0}^{\infty} P[H_A(\tau)=i\,/\,N_A(\tau)=n] \cdot P[N_A(\tau)=n]$$
$$= \sum_{n=1}^{\infty} \binom{n}{i} p_A^i\, q_A^{n-i}\, [F_n(\tau)-F_{n+1}(\tau)],\ i=1,2,\ \ldots\ldots$$

The probability $K(\tau;A)$ that A kills his passive target in time τ is obtained, from equations (14)−(15) and (16), as

$$(17)\quad K(\tau ; A) = \int_0^T k_A(t)\, dt$$

$$= \sum_{i=1}^{\infty} P[H_A(\tau) = i] \cdot c_{A_i} .$$

$$= \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \binom{n}{i} p_A^i\, q_A^{n-i} \,[F_n(\tau) - F_{n+1}(\tau)] \cdot [1-(1-c_A)^i]$$

where $c_A = c_{A_1}$.

Hence, we have

$$(18)\quad K^*(s; A) = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \binom{n}{i} p_A^i\, q_A^{n-i} \left[\frac{1}{s}(f_A^*(s))^n - \frac{1}{s}(f_A^*(s))^{n+1} \right] \cdot [1-(1-c_A)^i]$$

$$= \left(\frac{1-f_A^*(s)}{s}\right) \sum_{i=1}^{\infty} \left\{ (q_A f_A^*(s))^{-1} \left(\frac{q_A f_A^*(s)}{1-q_A f_A^*(s)} \right)^{i+1} \right\} \cdot \left(\frac{p_A}{q_A}\right)^i \left[1-(1-c_A)^i\right]$$

$$= \left(\frac{1}{s}\right) \cdot \left[\frac{p_A c_A f_A^*(s)}{1-(1-p_A c_A) f_A^*(s)} \right]$$

From equations (17)−(18), we find

$$k_A^*(s) = \frac{p_A\, c_A\, f_A^*(s)}{1-(1-p_A c_A)\, f_A^*(s)} .$$

Example II-1 (vital hit shot/negative exponential)

In this example, the probabilities $P(\tau ; A)$, $P(\tau ; AB)$ and $P(A)$ with vital hit shots r and k of equations (3), (5) and (7) are derived for negative exponential interfiring times.

Let $f_A(t) = r_A e^{-r_A t}$ and $f_B(t) = r_B e^{-r_B t}$, where r_A and r_B are rates of fire for A and B respectively. Then, we have

$$(19)\quad f_A^*(s) = \frac{r_A}{s+r_A} , \qquad f_B^*(s) = \frac{r_B}{s+r_B}$$

$$(20)\quad h_{A_r}^*(s) = \left(\frac{\lambda_A}{s+\lambda_A}\right)^r , \qquad h_{B_k}^*(s) = \left(\frac{\lambda_B}{s+\lambda_B}\right)^k$$

where $\lambda_A = r_A\, p_A$ and $\lambda_B = r_B\, p_B$.

From equations (4) and (20), we obtain

$$P^*(s\,;A) = (\frac{1}{s})\left\{\lambda_A^r\,\lambda_B^k\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{1}{s+z+\lambda_A}\right)^r\left(\frac{1}{-z+\lambda_B}\right)^k\frac{dz}{z}\right\}$$

where the contour integral has a pole of order k at $z=\lambda_B$.

The integral can be evaluated by expanding the integrand in Laurent series [4] and then finding the doefficient of $(\frac{1}{-z+\lambda_B})$.

we then have

$$(21)\quad P^*(s\,;A) = (\frac{1}{s})\sum_{y=0}^{k-1}\binom{y+r-1}{y}\left(\frac{\lambda_A}{s+\lambda_A+\lambda_B}\right)^r\cdot\left(\frac{\lambda_B}{s+\lambda_A+\lambda_B}\right)^y,$$

$$k^*(s\,;\bar{A}_r,B_k) = \sum_{y=0}^{k-1}\binom{y+r-1}{y}\left(\frac{\lambda_A}{s+\lambda_A+\lambda_B}\right)^r\left(\frac{\lambda_B}{s+\lambda_A+\lambda_B}\right)^y,$$

and

$$k(t\,;\bar{A}_r,B_k) = \sum_{y=0}^{k-1}\binom{y+r-1}{y}\frac{\lambda_A^r\,\lambda_B^y}{\Gamma(r+y)}\,t^{r+y-1}\,e^{-(\lambda_A+\lambda_B)t}$$

Hence,

$$(22)\quad P(\tau\,;A) = \int_0^{\tau} k(t\,;\bar{A}_r,B_k)\,dt$$

$$= \sum_{y=0}^{k-1}\binom{y+r-1}{y}\left(\frac{\lambda_A}{\lambda_A+\lambda_B}\right)^r\left(\frac{\lambda_B}{\lambda_A+\lambda_B}\right)^y\cdot\frac{\Gamma_\delta(r+y)}{\Gamma(r+y)}$$

where $\Gamma_\delta(\alpha) = \int_0^{\delta} y^{\alpha-1}e^{-y}\,dy$ and $\delta = (\lambda_A+\lambda_B)\,\tau$

By using the well - known relations

$$(22)\quad \sum_{y=0}^{k-1}\binom{y+r-1}{y}\eta^y = (1-\eta)^{-r}\cdot[1-I_\eta(k,r)]$$

and

$$(24)\quad \int_\eta^\infty e^{-x}x^L\,dx = L\,!\,P(X\leqq L)$$

$$= L\,!\,P(Y\geqq 2\eta)$$

where $I_\eta(k,r) = \frac{\Gamma(k)+\Gamma(r)}{\Gamma(k+r)}\int_0^\eta \rho^{k-1}(1-\rho)^{r-1}\,d\rho$,

X is a Poisson random variable with parameter η, and Y is a Chisquere pdf with degree $2L+2$, $P(\tau;A)$ of equation (22) can be rearranged to yield

$$(25)\quad P(\tau;A) = [1-I_\eta(k,r)] - L(\tau;A)$$

where

$$L(\tau;A) = (1-\eta)^r \sum_{y=0}^{k-1} \binom{y+r-1}{y} \eta^y e^{-(\lambda_A+\lambda_B)\tau} \cdot \sum_{i=0}^{r+y-1} \frac{\{(\lambda_A+\lambda_B)\tau\}^i}{i!}$$

and $\quad \eta = \dfrac{\lambda_B}{\lambda_A+\lambda_B}$

The probability of a draw can be written as

$$(26)\quad P(\tau;AB) = \left(\sum_{i=0}^{r-1} \frac{(\lambda_A\tau)^i e^{-\lambda_A\tau}}{i!}\right) \cdot \left(\sum_{j=0}^{k-1} \frac{(\lambda_B\tau)^j e^{-\lambda_B\tau}}{j!}\right) \quad \text{or}$$

$$= L(\tau;A) + L(\tau;B),$$

where $L(\tau;A)$ is given by equation (25) and $L(\tau;B)$ is obtained by interchanging λ_A and λ_B and r and k in $L(\tau;A)$.

We also have

$$(27)\quad P(A) = \lim_{T\to\infty} P(\tau;A) = 1-I_\eta(k,r)$$

which can also be obtained from equation (6) and (21).

We note that equation (21) coincides with Bhashyam's result [3].

Example II-2 (weighted average/negative exponential)

In this example, the probabilites $P(\tau;A)$ and $P(A)$ with weighted average values of equations (10) - (11) are derived for negative exponential interifiring times.

From equations (10) - (13) and (19), we have

$$(28)\quad P^*(s;A) = (r_A p_A c_A)(r_B p_B c_B) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{(s+z+r_A p_A c_A)(-z+r_A p_A c_B)} \cdot \frac{dz}{z}$$

$$= (\frac{1}{s}) \left[\frac{r_A\, p_A\, c_A}{s + r_A\, p_A\, c_A + r_B\, p_B\, c_B}\right]$$

which can be inverted to yield

$$(29) \quad P(\tau\,;A) = \frac{r_A\, p_A\, c_A}{r_A\, p_A\, c_A + r_A p_A\, c_A} \left[1 - e^{-(r_A\, p_A\, c_A + r_B\, p_B\, c_B)\, T}\right]$$

From this we also have

$$(30) \quad P(A) = \frac{r_A\ \ p_A\ c_A}{r_A\, p_A\, c_A + r_B\, p_B\, c_B}$$

whieh also coincides with Bhashyam′s result [3].

III. Multiple Hits with Limited Ammunition Supply

In this section we include the the kill effects of multiple hits in the fundamental duel when both duelists have limited ammunition supply and use single firing mode. Assumptions in Section II remain valid.

Here, we define that

α_i : the probability that A starts the duel with ammunition stock i rounds.

β_j is similarly defined for B.

Obviously

$$\sum_{i=0}^{\infty} \alpha_i = 1 \quad \text{and} \quad \sum_{j=0}^{\infty} \beta_j = 1\,.$$

$\phi_{A'_r(t)\,dt}$: the probability that A scores the vital hit shot on the passive target in $(t, t+dt]$ when the amount of initial ammunition stock (which is greater than or equal to r rounds) is limited.

ϕ_{A_r} : the probatility that the vital hit shot can not be obained due to ammunition shortage.

Then, we obtain

$$(31) \quad \phi_{A'_r(t)\,dt} = \sum_{i=r}^{\infty} \alpha_i \sum_{x=0}^{i-r} \binom{x+r-1}{x}\, p_A^{r}\, q_A^{x}\, f_A^{(r+x)}(t)\, dt$$

$$\phi_{A_r} = \sum_{i=0}^{\infty} \alpha_i \sum_{x=0}^{r-1} \binom{i}{x} p_A^x q_A^{i-x}.$$

Hence, A's probability density function $\phi_{A_r}(t)$ of the random time to obtain a sure kill at the r-th hit with limited ammunition can be expressed by techniques similar to Ancker [1], as

$$\phi_{A_r(t)\,dt} = \phi_{A'_r(t)\,dt} + \phi_{A_r} \cdot \delta(t-\infty)\,dt.$$

We then have

$$(32)\quad \phi^*_{A'_r(s)} = \sum_{i=r}^{\infty} \alpha_i \sum_{x=0}^{\infty} \binom{x+r-1}{x} p_A^r q_A^x (f_A^*(s))^{r+x}$$
$$= h^*_{A_r}(s) \cdot \left\{ \sum_{i=r}^{\infty} \alpha_i \left[1 - I_{(q_A f_A^*(s))}(i-r+1, r)\right] \right\}.$$

where $h^*_{A_r}(s)$ and $I_\eta(m, n)$ are given in equations (2) and (24) respectively. Similarly, we can obtain $\phi^*_{B'_k(s)}$ and ϕ_{B_k} as

$$(33)\quad \phi^*_{B'_k(s)} = h^*_{B_k}(s) \cdot \left\{ \sum_{j=k}^{\infty} \beta_j \left[1 - I_{(q_B f_B^*(s))}(j-k+1, k)\right] \right\}$$

and

$$(34)\quad \phi_{B_k} = \sum_{j=0}^{\infty} \beta_j \sum_{y=0}^{k-1} \binom{j}{y} p_B^y q_B^{j-y}.$$

A's winning probability $P(\tau; A)$ with limited duel time and limited ammunition supply can be expressed as

$$(35)\quad P(\tau; A) = \int_0^T \phi_{A_r}(t) \left[\int_t^\infty \phi_{B_k}(\tau)\,d\tau \right] dt$$
$$= \int_0^T \phi_{A'_r}(t) \left\{ \int_t^\infty \left[\phi_{B'_k}(\tau) + \phi_{B_k} \cdot \delta(\tau - \infty)\right] d\tau \right\} dt$$
$$= \int_0^T \phi_{A'_r}(t) \left[\int_t^\infty \phi_{B'_k}(\tau)\,d\tau \right] dt + \phi_{B_k} \cdot \int_0^T \phi_{A'_r}(t)\,dt.$$

With the same arguments leading to equations (4) and (7), we have

$$(36)\quad P^*(s; A) = \left(\frac{1}{s}\right) \left\{ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi^*_{A'_r}(s+z)\, \phi^*_{B'_k}(-z) \frac{dz}{z} + \phi_{B_k} \cdot \phi^*_{A'_r}(s) \right\}$$

and

$$(37)\quad P(A) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \phi^{*}_{A'_r}(z)\ \phi^{*}_{B'_k}(-z)\frac{dz}{z} + \phi_{B_k}\cdot(1-\phi_{A_r}).$$

Similar expressions for $P(\tau\,;B)$ and $P(B)$ can be obtained.

The probability $P(\tau\,;AB)$ of a draw due to duel time limitation and ammunition shortage is

$$\begin{aligned}(38)\quad P(\tau\,;AB) &= \left\{\int_T^{\infty}\phi_{A_r}(t)\,dt\right\}\cdot\left\{\int_T^{\infty}\phi_{B_k}(\tau)\,d\tau\right\}\\ &= \left\{\int_T^{\infty}\phi_{A'_r}(t)\,dt\right\}\cdot\left\{\int_T^{\infty}\phi_{B'_k}(\tau)\,d\tau\right\} + \phi_{A_r}\cdot\int_T^{\infty}\phi_{B'_k}(\tau)\,d\tau\\ &\quad + \phi_{B_k}\cdot\int_T^{\infty}\phi_{A'_r}(t)\,dt + \phi_{A_r}\cdot\phi_{B_k},\end{aligned}$$

whereas the probability $P(AB)$ of a draw due to ammunition shortage is

$$(39)\quad P(AB) = \phi_{A_r}\cdot\phi_{B_k}.$$

For the special case of fixed ammunition supply, we obtain a duel model as follows :

Let

$$(40)\quad \alpha_i = 1 \ \text{ for } i=m,\quad = 0 \ \text{ otherwise},\qquad \beta_j = 1 \ \text{ for } j=n,\quad = 0 \ \text{ otherwise}$$

where $m \geqq r$ and $n \geqq k$.

Substituting equation (40) into equations (31)-(34), we obtain

$$P^{*}(s\,;A) = \left(\frac{1}{s}\right)\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\phi^{*}_{A'_m}(s+z)\ \phi^{*}_{B'_n}(-z)\frac{dz}{z} + \phi^{*}_{A'_m}(s)\cdot\phi_{B_n}\right\},$$

$$(41)\quad P(A) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\phi^{*}_{A'_m}(z)\ \phi^{*}_{B'_n}(-z)\frac{dz}{z} + \phi_{B_n}(1-\phi_{A_m}) \ \text{ and}$$

$$P(AB) = \phi_{A_m}\cdot\phi_{B_n}$$

where

$$\phi^{*}_{A'_m}(s) = h^{*}_{A_r}(s)\cdot\left[1 - I_{(q_A f^{*}_A(s))}(m-r+1,\ r)\right],$$

$$\phi_{B'_n}(s) = h^{*}_{B_k}(s)\cdot\left[1 - I_{(q_B f^{*}_B(s))}(n-k+1,\ k)\right],$$

$$\phi_{A_m} = \sum_{x=0}^{r-1} \binom{m}{x} p_A^{\,x} q_A^{\,m-x},$$

and

$$\phi_{B_n} = \sum_{y=0}^{k-1} \binom{n}{y} p_B^{y} q_B^{n-y} .$$

Example Ⅲ-1 (vital hit shot/geometric/negative exponential)

The probabilities $P(r;A)$, $P(A)$ and $P(AB)$ with vital hit shots r and k are derived for geometric ammunition stocks and negative exponential inter-firing times.

Let

(42) $\alpha_i = (1-\alpha)\,\alpha^i$ and $\beta_j = (1-\beta)\,\beta^j$.

From equations (32) and (42), we can have

(43) $$\phi_{A_r'}^{*}(s) = (1-\alpha)\,(p_A f_A^{*}(s))^r \sum_{i=r}^{\infty} \alpha_i \sum_{x=0}^{i-r} \binom{x+r-1}{x} (q_A f_A^{*}(s))^x$$

$$= \left(\frac{\alpha p_A f_A^{*}(s)}{1-\alpha q_A f_A^{*}(s)}\right)^r$$

Furthermore, if $f_A(t) = r_A e^{-r_A t}$ and $f_B(t) = r_B e^{-r_B t}$, equation (43) is rearranged to yeild

(44) $$\phi_{A_r'}^{*}(s) = \left(\frac{\alpha r_A p_A}{s+r_A(1-\alpha q_A)}\right)^r$$

Substituting equation (42) into ϕ_{A_r} in equation (31), we also have

(45) $$\phi_{A_r} = (1-\alpha) \sum_{i=0}^{\infty} \alpha_i \sum_{x=0}^{r-1} \binom{i}{x} p_A^x q_A^{\,i-x}$$

$$= 1- \left(\frac{\alpha p_A}{1-\alpha q_A}\right)^r .$$

Similarly for B we have

(46) $$\phi_{B_k'}^{*}(s) = \left(\frac{\beta r_B p_B}{s+r_B(1-\beta q_B)}\right)^k$$

and

(47) $$\phi_{B_k} = 1- \left(\frac{\beta p_B}{1-\beta q_B}\right)^k$$

Hence, by substituting equations (44) – (47) into equations (35) – (39), the probabilities $P(\tau\,;A)$, $P(A)$, $P(\tau\,;AB)$ and $P(AB)$ can be obtained as follows.

First, we have

$$(48)\quad P^*(s;A) = \left(\frac{1}{s}\right)\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{\alpha r_A\, p_A}{s+z+r_A\,(1-\alpha q_A)}\right)^r\left(\frac{r_B\, p_B}{-z+r\,(1-\beta q_B)}\right)^k \frac{dz}{z} + \phi_{B_k}\cdot\left(\frac{\alpha r_A\, p_A}{s+r_A\,(1-\alpha q_A)}\right)^r\right\}$$

$$= \left(\frac{1}{s}\right)\left\{\left(\frac{\beta p_B}{1-\beta q_B}\right)^k\sum_{x=0}^{k-1}\binom{r+x-1}{x}\left(\frac{\theta_B}{s+\theta_A+\theta_B}\right)^x\cdot\left(\frac{\alpha r_A\, p_A}{s+\theta_A+\theta_B}\right)^r + \left[1-\left(\frac{\beta p_B}{1-\beta q_B}\right)^k\right]\left(\frac{r_A\, p_A}{s+\theta_A}\right)^r\right\}$$

where $\theta_A = r_A\,(1-\alpha q_A)$ and $\theta_B = r_B\,(1-\beta q_B)$.

$P^*(s\,;A)$ can be inverted to yield

$$(49)\quad P(\tau;A) = \left(\frac{\alpha p_A}{1-\alpha q_A}\right)^r\cdot\left\{\left(\frac{\beta p_B}{1-\beta q_B}\right)^k\cdot\left[1-I_z(k\,,r\,)\right] + \left[1-\left(\frac{\beta p_B}{1-\beta q_B}\right)^k\right]\right\} - A(\tau\,;A)$$

where

$$A(\tau\,;A) = \left(\frac{\beta p_B}{1-\beta q_B}\right)^k\cdot\left(\frac{\alpha r_A\, p_A}{\theta_A+\theta_B}\right)^r\cdot\sum_{x=0}^{k-1}\binom{r+x-1}{x}\left(\frac{\theta_B}{\theta_A+\theta_B}\right)^x\cdot\sum_{i=0}^{r+x-1}\frac{\{(\theta_A+\theta_B)\,\tau\,\}^i}{i\,!}\cdot e^{-(\theta_A+\theta_B)\,\tau} + \left[1-\left(\frac{\beta p_B}{1-\beta q_B}\right)^k\right]\cdot\left(\frac{\alpha p_A}{1-\alpha q_A}\right)^r\cdot\sum_{i=0}^{r-1}\frac{(\theta_A\,\tau\,)^i\, e^{-\theta_A\tau}}{i\,!},$$

$z = \dfrac{\theta_B}{\theta_A+\theta_B}$ and $I_z\,(k, r\,)$ is given by equation (23).

Hence, $P(A)$ can be expressed as equation (49) with $A(\tau\,;A) = 0$.

From equations (44) – (47) and (38), $P(\tau\,;AB)$ can be obtained as

$$(50)\quad P\,(\tau\,;AB) = \left\{\left(\frac{\alpha p_A}{1-\alpha q_A}\right)^r\sum_{i=0}^{r-1}\frac{(\theta_A\,\tau\,)^i}{i\,!}\,e^{-\theta_A\tau} + \left[1-\left(\frac{\alpha p_A}{1-\alpha q_A}\right)^r\right]\right\}\cdot$$

$$\left\{ \left(\frac{\beta p_B}{1-\beta q_B}\right)^k \sum_{j=0}^{k-1} \frac{(\theta_B \tau)^i}{i!} e^{-\theta_B \tau} + \left[1-\left(\frac{\alpha p_B}{1-\alpha q_B}\right)^k\right]\right\}$$

$$= A(\tau; A) + A(\tau; B) + \left[1-\left(\frac{\alpha p_A}{1-\alpha q_A}\right)^r\right] \cdot \left[1-\left(\frac{\beta p_B}{1-\beta q_B}\right)^k\right].$$

Obviously, $P(AB)$ becomes

$$(51) \quad P(AB) = \left[1-\left(\frac{\alpha p_A}{1-\alpha q_A}\right)^r\right] \cdot \left[1-\left(\frac{\beta p_B}{1-\beta q_B}\right)^k\right]$$

and equations (49) and (50) with $\alpha=\beta=1$ reduce to equations (25) and (26) respectively.

Example III-2 (weighted average/geometric/negative exponential)

Here we obtain the winning probability of a given side and the probability of draw with weighted averages $\phi_A(t)$ and $\phi_B(t)$ when random vital hit shot and random initial ammunition supply are geometric and interfiring time are negative exponential.

With arguments similar to those leading to equations (10)−(11), the weighted averages $\phi^*_{A'}(s)$ of $\phi_{A'_r}(s)$ and ϕ_A of ϕ_{A_r} can be obtained as

$$(52) \quad \phi^*_{A'}(s) = \sum_{r=1}^{\infty} \phi^*_{A'_r}(s) \left[c_A(1-c_A)^{r-1}\right]$$

$$= \frac{\alpha p_A c_A f_A^*(s)}{1-\alpha(1-p_A c_A) f_A^*(s)}$$

$$(53) \quad \phi_A = \sum_{k=1}^{\infty} \phi_{A_r}\left[c_A(1-c_A)^{r-1}\right] = \frac{1-\alpha}{1-\alpha(1-p_A c_A)}$$

where $\phi^*_{A'_r}(s)$ and ϕ_{A_r} are given in equations (43) and (45).

If $f_A(t) = r_A e^{-r_A t}$ and $f_B(t) = r_B e^{-r_B t}$, equation (52) can be rearranged to yield.

$$(54) \quad \phi^*_{A'}(s) = \frac{\alpha r_A p_A c_A}{s + r_A[1-\alpha(1-p_A c_A)]}$$

Similarly for B $\phi_B^*(s)$ and ϕ_B are obtained as

$$(55)\quad \phi_{B'}^*(s) = \frac{\beta r_B p_B c_B}{s + r_B [1-\beta(1-p_B c_B)]}$$

and

$$(56)\quad \phi_B = \frac{1-\beta}{1-\beta(1-p_B c_B)}$$

Then, $P^*(s;A)$, with the weighted averages in equations (53)−(56), is given by

$$(57)\quad P^*(s;A) = \left(\frac{1}{s}\right)\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\phi_{A'}^*(s+z)\,\phi_{B'}^*(-z)\frac{dz}{z} + \phi_B\cdot\phi_{A'}^*(s)\right\}$$

$$= \left(\frac{1}{s}\right)\left\{\left(\frac{\alpha r_A p_A c_A}{s + r_A[1-\alpha(1-p_A c_A)] + r_B[1-\beta(1-p_B c_B)]}\right)\cdot\right.$$

$$\left(\frac{\beta p_B c_B}{1-\beta(1-c_B p_B)}\right) + \left[\frac{1-\beta}{1-\beta(1-p_B c_B)}\right]\cdot$$

$$\left.\left[\frac{\alpha r_A p_A c_A}{s+r_A[1-\alpha(1-p_A c_A)]}\right]\right\},$$

and its inversion becomes

$$(58)\quad P(\tau;A) = \left(\frac{\beta p_B c_B}{1-\beta(1-p_B c_B)}\right)\left(\frac{\alpha r_A p_A c_A}{r_A[1-\alpha(1-p_A c_A)] + r_B[1-\beta(1-p_B c_B)]}\right)$$

$$\left\{1-\exp\left(-[r_A(1-\alpha(1-p_A c_A)) + r_B(1-\beta(1-p_B c_B))]\,\tau\right)\right\}$$

$$+\left(\frac{1-\beta}{1-\beta(1-p_B c_B)}\right)\cdot\left(\frac{\alpha p_A c_A}{1-\alpha(1-p_A c_A)}\right)\cdot$$

$$\left\{1-\exp\left(-r_A[1-\alpha(1-p_A c_A)]\,\tau\right)\right\}.$$

If the duel time is unlimited, equation (58) reduces to

$$(59)\quad P(A) = \left(\frac{\beta p_B c_B}{1-\beta(1-p_B c_B)}\right)\left(\frac{\alpha r_A p_A c_A}{r_A[1-\alpha(1-p_A c_A)]+r_B[1-\beta(1-p_B c_B)]}\right)$$

$$+\left(\frac{1-\beta}{1-\beta(1-p_B c_B)}\right)\left(\frac{\alpha p_A c_A}{1-\alpha(1-p_A c_A)}\right),$$

The probability of a draw becomes

$$(60)\quad P(AB) = \phi_A\cdot\phi_B = \left(\frac{1-\alpha}{1-\alpha(1-p_A c_A)}\right)\left(\frac{1-\beta}{1-\beta(1-p_B c_B)}\right)$$

Obviously, equation (59) reduces to equation (30) when $\alpha=\beta=1$.

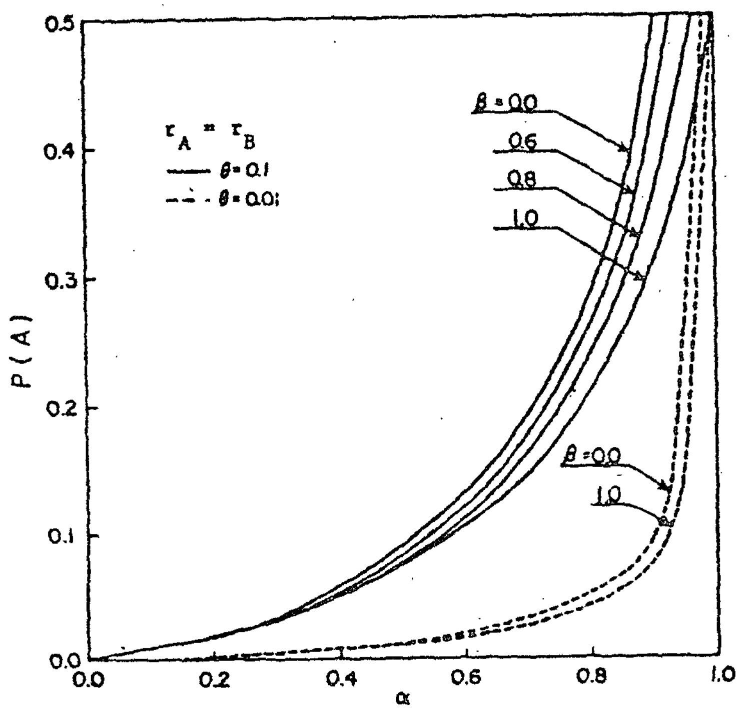

Fig. 1. P(A) with limited ammunition supply

Fig. 1 *depicts* $P(A)$ *in equation* (59) *with* $r_A = r_B$, $p_A c_A = p_B c_B = \theta$.
Two values of θ, $\theta = 0, 1$ and $\theta = 0.01$, are considered here.
It shows that A's winning probability increases for fixed α as β decreases and shifts downward as θ decreases. $P(A)$ with unlimited ammunition supply in this case has only a fixed value $\frac{1}{2}$, whereas $P(A)$ with limited ammunition supply has various values depending on the values on α and β.

Example Ⅲ-3 (weighted average/geometric/gamma with order 2)

Suppose that the interfiring times are random variables with pdf's $f_A(t) = 4r_A^2\, te^{-2r_A t}$ and $f_B(t) = 4r_B^2\, te^{-2r_B t}$. we then have

$$f_A^*(s) = \left(\frac{2r_A}{s+2r_A}\right)^2 \quad \text{and} \quad f_B^*(s) = \left(\frac{2r_B}{s+2r_B}\right)^2$$

From these and equations (52) − (53) we have

$$\phi_{A'}^*(s) = \frac{\alpha p_A c_A (2r_A)^2}{(s+2r_A)^2 - \alpha(1-p_A c_A)(2r_A)^2}$$

and

$$\phi_A = \frac{1-\alpha}{1-\alpha(1-p_A c_A)}$$

we also have

$$\phi_{B'}^*(s) = \frac{\beta p_B c_B (2r_B)^2}{(s+2r_B)^2 - \beta(1-p_B c_B)(2r_B)^2}$$

and

$$\phi_B = \frac{1-\beta}{1-\beta(1-p_B c_B)} \cdot$$

Hence, A′s winning probability $P(A)$ with the above weighted average values can be written as

$$(61) \quad P(A) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \phi_{A'}^*(s)\, \phi_{B'}^*(-s)\, \frac{ds}{s} + (1-\phi_A)\cdot\phi_B$$

$$= \alpha p_A c_A (2r_A)^2 \cdot \beta p_B c_B (2r_B)^2 \cdot \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} [(s+2r_A)^2 - \alpha(1-p_A c_A)(2r_A)^2]^{-1} \cdot [(-s+2r_B)^2 - \beta(1-p_B c_B)(2r_B)^2]^{-1} \cdot \frac{ds}{s} + \left(\frac{1-\beta}{1-\beta(1-p_B c_B)}\right)\left(\frac{\alpha p_A c_A}{1-\alpha(1-p_A c_A)}\right).$$

Since the contour integral of the above equation has two poles at $s = 2r_B(1+\sqrt{\beta(1-p_B c_B)})$ and $s = 2r_B(1-\sqrt{\beta(1-p_B c_B)})$ it can be evaluated by summing two residues at each pole.

The result becomes

$$(62)\quad P(A) = \left(\frac{\alpha\beta\, p_A\, p_B\, c_A\, c_B\, r_A^2}{1-\beta(1-p_B\, c_B)}\right)\left(\frac{M}{N}\right) + \left(\frac{1-\beta}{1-\beta(1-p_B\, c_B)}\right)\left(\frac{\alpha\, p_A\, c_A}{1-\alpha(1-p_A\, c_A)}\right)$$

where

$$M = r_A^2\,[1-\alpha(1-p_A\, c_A)] - r_B^2\,[1-\beta(1-p_B\, c_B)] + 4\, r_B\,(r_A + r_B),$$

and

$$N = \left\{r_A^2\,[1-\alpha(1-p_A\, c_A)] - r_B^2\,[1-\beta(1-p_B\, c_B)]\right\}^2 + 4\, r_A\, r_B\,(r_A + r_B)\cdot \left\{r_A\,[1-\alpha(1-p_A\, c_A)] + r_B\,[1-\beta(1-p_B\, c_B)]\right\}.$$

we also have

$$(63)\quad P(AB) = \frac{(1-\alpha)(1-\beta)}{[1-\alpha(1-p_A\, c_A)]\,[1-\beta(1-p_B\, c_B)]}$$

which coincides with Ancker's result [1] if $c_A = c_B = 1$.

IV. Discussions

In this paper, the "fundamental duel" studied by Williams and Ancker is extended to include various limiting factors such as a sure kill by a multiple hits, conditional kill probability on hit, fixed duel time, limited amount of ammunition, etc.

The results of this study may be utilized in evaluating weapon, combat and combat– support capabilities, and in designing optimal levels of weapon effectiveness parameters. The stochastic duel models developed here for the case of single firing mode may also be extendeded to the case of multiple firing modes ; pattern firing [7], salvo firing and dispenser firing.

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