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AN APPLICATION OF PROJECTIVE REPRESENTATION TO FINITE GROUPS.

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ABSTRACT

群의 表現論은 有限群의 性質究明에 많이 利用되여진다. 이 글에서도 어떤條件下에서 p-Sylow 群 S_p가 order p을 갖는 두개의 巡回群의 곱으로 分解됨을 表現論에 의하여 證明을 試圖한다(本文 Thorem 7).

Introductions

The group representation theory has been applied to the group theory by many mathematicians. In this note, we shall try to find some properties of a finite group applying the projective representation theory. In fact, even if we know that a finite group S_p with order P^2(p is prime) is abelian, we don't know whether G is a cyclic group or not. In this note, it will be proved that under some conditions S_p has a direct product decomposition $C_p \times C_p$, where C_p is a cyclic group with order P (Theorem 7).

DEFINITION 1. we put

C the field of all complex numbers, $C^* = C - \{0\}$

For a finite group G, if a function $\alpha : G \times G \to C^*$ satisfies the condition

$$\alpha(xy, z)\alpha(x, y) = \alpha(y, z)\alpha(x, yz),$$

then, it is called a factor set of G. The projective group algebra $C(G)_\alpha$ of G is defined by

(i) the underlying set of $C(G)_\alpha$ is equal to the underlying set of the group algebra $C(G)$. That is, $C(G)_\alpha$ is a C-algebra with bases $\{U_x | x \in G\}$.

(ii) the multiplication in $C(G)_\alpha$ is given by

$$U_x \cdot U_y = \alpha(x, y) U_{xy}$$

for all $U_x, U_y \in C(G)_\alpha$.

DEFINITION 2. Let G be a finite group with center Z. If there is an irreducible complex character x of G such that $x(1)^2 = |G/Z|$, then G is said to be central type.

LEMMA 3. Let H be a finite group with a factor set α such that $Z(\mathbf{C}(H)_\alpha) = C$.

Then there is a group G of central type with $G/Z(G)\cong H$, where $Z(G)$ is the center of G.

PROOF. Let G be a representation group of H. That is,

(i) the sequence of goups

$$1 \longrightarrow Z(G) \longrightarrow G \longrightarrow H \longrightarrow 1$$

is exact.

(ii) ervery projective representation T^* of H is lifted to an ordinary representation T of G. By Maschke's theorem ([2]) $C(H)$ is complete reducible, so we can write such that

$$C(H)=\mathrm{Hom}_C(V_1,V_1)\oplus\cdots\oplus \mathrm{Hom}_C(V_n,V_n),$$

where $V_i(i=1,2,\cdots,n)$ is an irreducible $C(H)$-module. Of course, each $\mathrm{Hom}_C(V_i,V_i)$ $(i=1,2,\cdots,n)$ is a $C(H)_\alpha$-module, and also the following holds

$$C(H)_\alpha=\mathrm{Hom}_c(V_1,V_1)\oplus\cdots\oplus \mathrm{Hom}_C(V_n,V_n),$$

Since $Z(C(H)_\alpha)=C$, if we put

$$\dim_c(V_1)\leq\min\{\dim_c(V_i)\,|\,i=2,3,\cdots,n\},$$

then

$$C(H)_\alpha=\mathrm{Hom}_c(V_i,V_i)$$

(Note that $Z(\mathrm{Hom}_c(V_i,V_i))=CI$, where I is the identity matrix).

We assume that $T:G\to V_1$ affords the irreducible character χ of G_i. Then $\chi(1)_2=c\dim_c(V_1))^2$.

By our Definition 1 $\dim_c(C(H)_\alpha)=\dim_c(C(H)=|H|$

($|H|$ is the order of H). Therefore,

$$\chi(1)^2=(\dim_c(V_1))^2=H|=|G/Z(G)|,$$

and thus G is a group of central type.

LEMMA 4. If G is a group of central type, then there is a factor set α of $\bar{G}=G/Z(G)$ such that $Z(C(\bar{G})_\alpha)=C$.

PROOF. Let χ be an irreducible character of G such that $\chi(1)^2=|G/Z(G)|$.

By the same way in the proof of Lemma 2, there is a factor set α of $\bar{G}$ such that an irreducible projective representation.

$$T^*:\ G\longrightarrow GL(V)$$

which is lifted to an irreducible ordinary representation

$$T:\ G\longrightarrow GL(V)$$

such that T affords the irreducible character χ of G.

Since $|\bar{G}|=\chi(1)^2=(\dim_c(V))^2$ we have

$$|\bar{G}| \simeq \mathrm{Hom}_c(V,V).$$

On the other hand, since $dim_c\ (C(\bar{G})_\alpha) = \dim_c(C(\bar{G})) = |\bar{G}|$, we have

$$C(\bar{G})_\alpha \cong \mathrm{Hom}_c(V,V).$$

$Z(\mathrm{Hom}_c(V,V)) \cong C$ implies that $Z(C(\bar{G})_\alpha) = C$.

THEOREM 5. Let G be a finite group and $\bar{G} = G/Z(G)$. Then G is a group of central type if and only if there is a factor set α of $\bar{G}$ such that the center of $C(\bar{G})_\alpha$ is equal to C.

PROOF. This result is clear by Lemma *3* and *4*.

LEMMA 6. Let G be a finite group and put $G/Z(G) = \bar{G}$. Then there is a factor set α of $\bar{G}$ such that $C(\bar{G})_\alpha = C$ if and only if for any Sylow p-subgroup S_p of G, $Z(S_p) = S_p \cap Z(G)$ and $C(\bar{S}_p)_\alpha$ has center C when $\alpha|\bar{S}_p$ is regarded as a factor set of $\bar{S}_p$, where $\bar{S}_p = S_p/Z(S_p)$.

PROOF. By Theorem 5, $Z(C(\bar{G})_\alpha) = C$ implties that G is a group of central type, and thus there is an irreducible character χ of G such that $\chi(I)^2 = |\bar{G}| = |G/Z(G)|$. Then, we have a linear character ϕ of $Z(G)$ such that $\phi^G = \chi(I)\chi$ ([1]). Let S_p be a Sylow p-subgroup of G and let $R = \langle S_p, Z(G) \rangle$ be the subgroup of generated by S_p and $Z(G)$.
Take an irreducible character γ of R satisfying

$$\phi^R = \gamma + (\text{other irreducible R-representations}).$$

Let $T: Z(G) \longrightarrow W$ afford the irreducible character ϕ.
Then ϕ^R is afforded by a representaion

$$T^R:\ R \longrightarrow GL(C(R) \otimes_{C(Z(G))} W),$$

where

$$C(R) \otimes_{C(Z(G))} W = (S_p/S_p \cap Z(G) \otimes_C W.$$

Therefore γ is afforded by a representation $R \longrightarrow GL$ (an irreducible $C(R)$-submodule of

$$(S_p/S_p \cap z(G) \otimes_C W) = GL(V).$$

Hence $\gamma|S_p$ is an irreducible character of S_p.
In this case,

$\chi(I)\chi = \phi^G = (\phi^R)^G = \gamma^G + (\text{other irreducible R-representations})^G$, so there is a positive integer k such that $I^G = k\chi$. In fact, the representation module of γ^G is equal to

$$C(G) \otimes C(R)^V = (G/R) \otimes_C V,$$

and thus

$$\gamma^G(1)=\dim_C(G/R)\otimes_C V,)=\dim_C(V)\cdot|G/R|=\gamma(1)\cdot|G/R|.$$

Hence we have

$$k\chi(1)=\gamma(1)\cdot|G/R|$$

$$\therefore\ k=\frac{|G/R|\cdot\gamma(1)}{\chi(1)} \qquad (※)$$

Noting that $\gamma(1)|S_p$ (for γ is an irreducible character of S_p), we see that there is a positive integer a such that $\gamma(1)=P^a$. $\cdot$ Since S_p is a Sylow p-subgroup of G, $P^b\nmid|G/R|$ for all $b=1,2,3,\cdots$.

On the other hand, since

$$\chi(1)^2=|G/Z(G)|=|G/R|\cdot|R/Z(G)|,$$

if we put for some positive integers m,n,l, and γ such that

$$|R|=|S_p|m,\ |S_p|=|S_p\cap Z(G)|P^n$$
$$|Z(G)|=|S_p\cap Z(G)|l,\ \ |G|=|R|r,$$

then,

$$\chi(1)^2=\frac{mr}{l}\cdot p^n.$$

where $\frac{mr}{l}$ is not any multiple of p. Thus, we get a positive integer

$$\chi(1)=\left(\frac{mr}{l}\right)^{\frac{1}{2}}p^{\frac{n}{2}}.$$

From (※),

$$\text{a positive integer}=k=\left(\frac{lr}{m}\right)^{\frac{1}{2}}\gamma(1)/p^{\frac{n}{2}},$$

and thus we have

$$\gamma(1)^2=p^n=|S_p/S_p\cap Z(G)|$$

(Note that $\gamma(1)=p^a$). Furthermore,

$$\gamma(1)^2\leq|S_p/Z(S_p)|\leq|S_p/S_p\cap Z(G)|=p^n=\gamma(1)^2,$$

and thus $Z(S_p)=S_p\cap Z(G)$ and S_p is a group of central type.

By Theorem 5, if we set $\bar{S}_p=S_p/Z(S_p)$, then $C(\bar{S}_p)_\alpha$ has center C. Conversely, we assume that for any Sylow p-sulgroup S_p of G, $Z(S_p)=S_p\cap Z(G)$ and $C(\bar{S}_p)_{\alpha p}$ has center C, where $\bar{S}_p=S_p/Z(S_p)$ and α_p is a factor set of $\bar{S}_p$.

Let $S_p,\cdots,S_r$ be a collection of Sylow subgroups for G, where each prime is a divisor of $|G|$, By Theorem 5, there are irreducible characters $\chi_p,\cdots,\chi_r$ such that for each S_p $\chi_p(1)^2=|S_p/Z(S_p)|=|\bar{S}_p|$. In this case,

$$G=S_p\times\cdots\times S_r,\ \ Z(G)=Z(S_p)\times\cdots\times Z(S_r),$$

and for each $Z(S_p)$ there is a linear character ϕ_p of $Z(S_p)$ such that $\phi_p^{S_p}=\chi_p(1)\chi_p$ (or $\chi_p|Z(S_p)=\chi_p(1)\phi_p$).

Then, we also have a linear character

$$\phi=\phi_p\times\cdots\times\phi_r$$

of $Z(G)$. Note that $\phi|Z(S_p)=\phi_p$. For an irreducible character χ of G there is a positive integer m such that

$$\phi^G=m\chi+\cdots\cdots.$$

Put $R_p=\langle S_p, ZG\rangle$ and $\phi^{R_p}=\chi_p(1)\chi'$, where χ' is an irreducible character of R_p such that $\chi'_p|S_p=\chi_p$.

Thus,

$$(\phi^{R_p})^G=\phi^G=\chi_p(1)\chi'^G=m\chi+\cdots.$$

Hence, since χ is an irreducible we have $\chi_p(1)|m$.

In conseguence,

$$|G/Z(G)|^{\frac{1}{2}}=\{|S_p/Z(S_p)|\cdots|S_r/Z(S_r)|\}^{\frac{1}{2}}=\chi_p(1)\cdots\cdots\chi_r(1)$$

and

$$\chi_p(1)\cdots\chi_r(1)|m.$$

Since $m=(\phi^G,\chi)G\leq\chi(1)$ and $\chi(1)||G/Z(G)|=\{\chi_p(1)\cdots\chi_r(1)\}^2$, we obtain $\chi(1)^2=m^2=\{\chi_p(1)\cdots\chi_r(1)\}^2=|G/Z(G)|$.

This means that G is a group of central type.

By Theorem 5, if we set $\bar{G}=G/Z(G)$ then $C(G)_\alpha$ has center C, where α is a factor set of $\bar{G}$ induced from the factor sets α_p of $\bar{S}_p$.

If G is a group of central type with $G/Z(G)$ abelian, then there is an abelian group H such that $G/Z(G)\cong H\times H$.

Using this fact we shall prove the following main theorem.

THEOREM 7. For a finite group G we put $\bar{G}=G/Z(G)$.

We assume that $|\bar{G}|=p_1^2\cdots\cdots p_n^2$, where the $p_i(i=1,\cdots,n)$ are distinct primes. If for a factor set α of $\bar{G}$ $C(\bar{G})_\alpha$ has center C then $\bar{G}=C_{p_1}\times C_{p_1}\times\cdots\times C_{p_n}\times C_{p_n}$, where $C_{p_i}(i=1,\cdots,n)$ are cyclic group with order p_i.

PROOF. By our assumption and Lemma 6 (or Th. 5) G is a group of central type, and thus a Sylow p_i-*subgroup* S_{p_i} of G is a group of central type as in the proof of Lemma

6. Then $|S_{p_i}/Z(S_{p_i})|=p_i^2$ and thus $S_{p_i}/Z(S_{p_i})$ is abelian. Therefore, there exists an abelian group H_{p_i} such that $S_{p_i}/Z(S_{p_i})=H_{p_i}\times H_{p_i}$.

Since $|H_{p_i}|=p_i$ H_{p_i} is a cyclic group C_{p_i}, and thus $S_{p_i}/Z(S_{p_i})=C_{p_i}\times C_{p_i}$. Hence $\bar{G}=C_{p_1}\times C_{p_1}\times\cdots\times C_{p_n}\times C_{p_n}$.

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