HONAM MATHEMATICAL JOURNAL
Volume 1, Number 1, July 1979

A STUDY ON PARACOMPACTNESS IN CONVEXITY

HONG-GI, KIM

ABSTRACT

본 논문에서는 Banach 공간의 Convex subset에서 보존되는 몇가지 위상적 성질을 조사하고 정리 2.3을 증명하였다.

Introduction

In this paper some topological properties preserved in convex subset of a Banach space are investigated and prove if X be paracompact, Y a Banach space with metric ρ, $\alpha < 1$, $P_\alpha(Y)$ the family of closed α-paraconvex, nonempty subsets of Y and ϕ; $X \to P_\alpha(Y)$ lower semi-continuous, then,

(a) there exists a selection ϕ,

(b) for some $r > 0$, there exists a continuous $g: X \to Y$ such that $\rho(g(x), \phi(x)) < r$ for all $x \in X$, then there exists a selection f for ϕ such that $\rho(g(x), f(x)) < \tilde{\alpha} r$, where $\tilde{\alpha} = 1 + \sum_{0}^{\infty} \alpha^i$.

(I)

The concept of paracompactness has been studied mainly for the purpose of metrizability. In particular the follwing characterization is well known (3)

"If X is a regular space, the following statements are equivalent".

a) The space X is paracompact

b) Each open cover of X has a locally finite refinement

c) Each open cover of X is even

d) Each open cover of X has a closed locally finite refinement

e) Each open cover of X has an open σ-discrete refinement

f) Each open cover of X has an open σ-locally finite refinement.

However, the usefulness of the concept of paracompactness easily extended in the convexity. In this paper some topological properties preserved in convex subset of a Banach space are investigated. In the begining some definitions and lemmas which will be used in the sequel are introduced.

1.1 DEFINITION: A locally covex space which is metrizable and complete is called a *Frechet space* and linear space with a norm which is complete relative to the norm topology is called a *Banach space.*

1.2 DEFINITION: ϕ is lower semicontinuous if $\{x\in X \mid \phi(x)\cap U\neq\phi\}$ is open in X for every open $U\subset Y$. A selection for ϕ is a continuous $f:X\to Y$ such that $f(x)\in\phi(x)$ for every $x\in X$.

The following theorems were introduced by *R.* Arens (1).

1.3 THEOREM: If X is paracompact, A a closed subset of X and a closed, convex subset C of a Banach space Y, then every continuous g: $A\to C$ can be extended to a continuous f: $X\to C$.

1.4 DEFINITION: Let E be a normed linear space with metric ρ and let α be a numbers such that $0\leq\alpha\leq 1$. Then a subset P of E is α-convex if, whenever $p\in E$ and $r>0$ are such that $\rho(p.P)<r$, then $\rho(q,P)\leqq dr$ for all $g\in conv$ $(Sr(p)\cap P)$ where $Sr(x)$ denotes the open r-sphere about x, and *conv* (A) denotes the convex hull of A.

1.5 DEFINITION: The set P is called paraconvex if it is α-paraconvex for some $\alpha<1$.

REMARK: It is clear that a closed set is 0-paraconvex iff it is convex. *V.L* Klee showed (2) that every subset of E is 1-paraconvex iff E is either an inner product space or two-dimensional.

Since paraconvexity is not a very intuitive concept, the examples of subsets of the Euclidean plane all of them compact, one-dimensional absolute retracts are as follows:

1.6 EXAMPLE: The letters V,X,Y and Z a circular arc subtending an angle$<\pi$., are paraconvex. The sharper the angle of the V, and the closer to π the angle subtended by the arc, the closer to 1 and must take α for these sets to be α-paraconvex.

1.7 EXAMPLE: The letter U and a circular are subtending an angle$\geq\pi$, are not paraconvex. In the case of the U, the midpoint p of the line segement joining the end points of the U violates the definition of α-paraconvexity for any $\alpha<1$; In the case of the circular arc, the same difficulty occurs when one takes p the center of the circle.

(II)

We obtain following result as a generalization of. 1.4

2.1 THEOREM: If X is paracompact, Y a Banach space and $C(Y)$ the family of closed, convex, non-empty subset of Y, then every lower semi-continuous φ: $X\to e(Y)$ admits a selection.

RROOF: Let X, $A\subset X$, Y, $C\subset Y$ and g; $A\to C$ be as in Theorem 1.4. Define x, $X\to C(Y)$ by

$$x(x)=\{g(x)\} \text{ if } x\in A,\ \chi(x)=C \text{ if } x\in X-A$$

Thus χ is lower semi-continuous and a selection for χ is required extension of g.

The following is equivalentt to 2.1. We denote the following of non-empty subset of Y by Z^Y.

2.2 THEOREM: Let X be paracompact, Y a Banach space, and $\phi:X\to Z^Y$ lower semi-continuous. Then there exists a continuous $f:X\to Y$ such that $f(x)\in(conv\ (\varphi(x)))$ for every $x\in X$.

PROOF: The theorem 2.1 obviously implies 2.2 and the converse follows from the fact that the lower semi continuity of ϕ implies that of χ defined by $\chi(x)=\varphi(x)$.

Furthermore, using the result of 2.2, we obtain following theorem.

2.3 THEOREM: Let X be paracompact, Y a Banach space with metric ρ, $\alpha<1$, $P_\alpha(Y)$ the family of closed, α-paraconvex, non-empty subsets of Y, and $\varphi;X\to P_\alpha(Y)$ lower semi-coninuous. Then

(a) there exists a selection for φ

(b) for some $r>0$, there exists a continuous g; $X\to Y$ such that $\rho(g(x),f(x))<r$ for all $x\in X$, then there exist a selection f for ϕ such that $\rho(g(x), f(x))<\tilde{\alpha}r$, where $\tilde{\alpha}=1+\sum_0^\infty \alpha^i$.

PROOF: Let $r>\alpha$ such that $\sum_0^\infty Y^i<\tilde{\alpha}$. By induction, we shall define a sequence of continuous functions $f_n, X\to Y$ $n=0,1\cdots\cdots$ with $f_0=g$, such that, for all n and all $n\in X$,

(1) $\rho(f_n(x),\ \phi(x))<\Upsilon^n r$

(2) $\rho(f_n(x),\ f_{n+1}(x))\leqq\Upsilon^n r$

This sequence of continuous functions is uniformly Cauchy, and hence has a continuous limit f. This f is a selection for ϕ by (1) and $\rho(g(x), f(x))<\tilde{\alpha}r$ by (2).

Let $f_0=g$, suppose f_1, $f_2\cdots\cdots f_n$ have been constructed and let us construct f_{n+1}. Define $\phi_{n+1}:X\to Z^Y$ by $\phi_{n+1}(x)=S\Upsilon^n r\ (f_n(x))\cap\phi(x)$: then ϕ_{n+1} is never empty, by the inductive assumption on f_n and ϕ_{n+1} is lower semi continuous. Hence, by Theorem 2.2, there exists a continuous f_{n+1}: $X\to Y$ such that $f_{n+1}(x)\in(conv\ (\phi_{n+1}(x)))$ for every $x\in X$. This f_{n+1} clearly satisfies (2) and it satisfies (1) because each $\phi(x)$ is α-paraconvex, where

$$\rho(f_{n+1}(x),\ \phi(x)) \leq \alpha\gamma^n r < \gamma^{n+1} r$$

for all $x \in X$ and (b) is proved. The proof (a) is similary proved as (b).

2.4 COROLLARY: If X is paracompact, A a closed subset of X and P a closed paraconvex subset of a Banach space, then every continuous g: $A \to P$ can be extended to a continous f; $X \to P$

Referances

1. R. Arens,; *Extensions of Functions in fully Normal Spaces*, Pacific J. Math., (2) 1952, 11～23.
2. Gillman-Jerison; *Rings of continuous Functions*, D. Van Norstrand Comp. 1960.
3. J. L. Kelley; *General Topology*, D. Van Nostrand Comp. 1955.
4. J. L. Kelley-I. Nanioka; *Linear Topological Spaces*, D. Van Norstrand Comp. (1963).
5. V.L. Klee; *Circumsphere and Inner Product*, Math. Scand. 8, 1960.
6. E. Michael; *Convex Structres and Continuous Selection*, Canadian J. Math. 11, 1959.

Jeju University