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Schur Multipliers and Cohomology of Finite Groups.

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ABSTRACT

G를 유한군으로, C를 모든 복소수체로 가정하고, V를 C상에서의 유한차원 벡터 공간이라 하자. V상에서의 G의 사영적 표시는, x, $y\in G$이고 α: $G\times G\to C$를 Facto set이라 할 때

$$T(x)T(y)=T(xy)\alpha(x,y)$$

이 되는 함수 T: $G\to GL(V)$를 말한다.

본 논문의 목적은 군에 대한 Extension theory를 사용해서, G상의 factor set들의 동치류들은 G의 Second Cohomology group과 동형이라는 것을 증명하는 것이다.

Introduction

Throughout this note, We assume that G is a finite Group and C is the field of all complex numbers. Let V be a finite dimensional vector space over C. A projective representation of G on V is a function T: $G\to GL(V)$ such that

$$T(x)\ T(y)=T(xy)\ \alpha\ (x,y,)$$

where $X, y\in G$ and $\alpha: G\times G\to C$ which is called the Factor set of T.
The purpose of this note is to prove by using "Extention theory of groups" that the equivalence classes of factor sets on G is isomorphic to the second cohomology group of G.

§ 1. The Schur Multipliers

DEFINITION 1. Let us put $C^*=C-\{0\}$. If a function α: $G\times G\to C^*$ satisfies

$$\alpha(x,yz)\ \alpha(y,z)=\alpha(x,y)\ \alpha\ (xy,z)$$

for all $x,y,z\in G$ then α is called a factor set of G. For two factor sets α and β of G, if there is a function $C: G\to C^*$ such that $\alpha(x,y)=\beta(x,y)\ C(x)\ C(y)\ C(xy)^{-1}$ for all $x,y\in G$, then α and β are said to be equivalent, Written $\alpha\sim\beta$.
It is easy to prove that $\sim$ is an equivalence relation.

LEMMA 2. If α is the factor set of a projective representation of G, then α is a factor set of G.

PROOF For $x, y, z \in G$, since $T(x) \in GL(v)$ we have

$$T(x)[T(y)T(z)] = [T(X)T(y)]T(z).$$

Since $T(x)T(y) = \alpha(x,y)T(xy)$,

$$T(x)[T(y)T(z)] = T(x)\alpha(y,z)T(yz) = \alpha(x,yz)\alpha(y,z)\ T(xyz)$$
$$[T(x)T(y)]T(z) = \alpha(x,y)T(xy)T(z) = \alpha(x,y)\alpha(xy,z)T(xyz)$$
$$\therefore\ \alpha(x,yz)\alpha(y,z) = \alpha(x,y)\alpha(xy,z).$$

Let T_1; $G \to GL(V_1)$ and T_2: $G \to GL(V_2)$ be projective representations of G. If there is an C-Vector space isomorphism f: $V_1 \to V_2$ such that $T_1(x) = C(x)\ f^{-1}T_2(x)f$ for some $c(x) \in C$ and all $x \in G$, then T_1 and T_2 are said to be equivalent.

LEMMA 3. If two projective representations T_1 and T_2 of G are equivalent, then their factor sets are also equivalent.

PROOF. We compute $T_1(x)\ T_2(y)$ in two way;

$$T_1(x)T_1(y) = \alpha_1(x,y)T_1(xy) = \alpha_1(x,y)c(xy)\ f^{-1}T_2(xy)f,$$

and

$$T_1(x)T_1(y) = C(x)f^{-1}T_2(x)f \cdot T_2(y)\ fC(y)$$
$$= C(x)C(y)f^{-1}T_2(x)T_2(y)f$$
$$= C(x)C(y)\alpha_2(x,y)\ f^{-1}T_2(xy)f.$$

Where $\alpha i (i = 1,2)$ is the factor set of Ti and f is a C-Vector space isomorphism, Therefore,

$$\alpha_1(x,y) = \alpha_2(x,y)C(x)C(y)C(xy)^{-1}.$$

Let M_G be the set of all factor sets of G. For $\alpha, \beta \in M_G$
We define $(\alpha\beta)(x,y) = \alpha(x,y)\beta(x,y)$ for all $x, y \in G$ and define $\alpha^{-1}(x,y) = \alpha(x,y)^{-1}$.
Then, if $\alpha \sim \alpha^1$ and $\beta \sim \beta^1$ we have $\alpha\beta \sim \alpha^1\beta^1$ and $\alpha^{-1} \sim (\alpha^1)^{-1}$.
They are easily proved as follows:

$$\left.\begin{array}{l}\alpha \sim \alpha^1 \Rightarrow \alpha(x,y) = \alpha^1(x,y)C_1(x)C_1(y)C_1(xy)^{-1} \\ \beta \sim \beta^1 \Rightarrow \beta(x,y) = \beta^1(x,y)C_2(x)C_2(y)C_2(xy)^{-1}\end{array}\right\} \Rightarrow$$
$$(\alpha\beta)(x,y) = \alpha(x,y)\beta(x,y) = \alpha'(x,y)\beta'(x,y)C_1(x)C_2(x)C_1(y)C_2(y)\ C_1(xy)^{-1}C_2(xy)^{-1}$$
$$= (\alpha^1\beta^1)(x,y)C(x)C(y)C(xy)^{-1}, \text{ where } C(x) = C_1(x)C_2(x).$$

Therefore, $\alpha\beta \sim \alpha^1\beta^1$. Also,

$$\alpha \sim \alpha^1 \Rightarrow \alpha(x,y) = \alpha'(x,y)C(x)C(y)C(xy)^{-1}$$
$$\Rightarrow \alpha^{-1}(X,y) = \alpha(x,y)^{-1} = \alpha^1(x,y)^{-1}C(x)^{-1}C(y)^{-1}C(xy)$$
$$= \alpha'^{-1}(x,y)C^1(x)C^1(y)C^1(xy)^{-1},$$

Where $C^1(x) = C(x)^{-1}$, Hence, We have $\alpha^{-1} \sim (\alpha')^{-1}$. Let us Put

$M_G = M_G/\sim$,

Which is Called the Schur multiplier of G.

LEMMA 4. M_G is a finite group.

PROOF, Let $|G|$ be the number of all elements in G.

At first, we shall prove that for every $\{\alpha\}\in M_G \{\alpha\}^{|G|}=1$.

Let α be a representative of $\{\alpha\}$.

For $x\in G$ define

$$\varphi\alpha(x)=\prod_{z\in G}\alpha(x,z).$$

Since $\alpha(x,y)\alpha(xy,z)=\alpha(x,yz)\alpha(y,z)$ for $x,y,z,\in G$

We have

$$\alpha(x,y)=\frac{\alpha(x,yz)\alpha(y,z)}{\alpha(xy,z)},$$

and thus

$$\alpha(x,y)^{|G|}=\prod_{z\in G}\frac{\alpha(x,yz)\alpha(y,z)}{\alpha(xy,z)}=\frac{\varphi_\alpha(x)\varphi_\alpha(y)}{\varphi_\alpha(xy)},$$

Where $\bigcup_{z\in G} yz=G$. Therefore, We have

$$\alpha(x,y)^{|G|}=1(x,y)\varphi_\alpha(x)\varphi_\alpha(y)\varphi_\alpha(xy)^{-1},$$

and thus

$$\{\alpha\}^{|G|}=\{1\}.$$

Where 1: $G\times G\to C^*$ is defined by $1(x,y)=1$ for all $x,y\in G$. Next, we shall prove that if $\{\alpha\}^e=\{1\}(e\leq|G|)$

there is a factor set α' of G suchthat $\alpha'(x,y)^e=1$ and $\alpha'\in\{\alpha\}$. Since $\{\alpha\}^e=\{1\}$ there is a function $a:G\to C^*$ such that

$$\alpha(x,y)^e=a(x)a(y)a(xy)^{-1}.$$

We define a function b: $G\to C^*$ such that $b(x)^e a(x)=1$ for all $x\in G$.

Define $\alpha'\in M_G$ by

$$\alpha'(x,y)=\alpha(x,y)b(x)b(y)b(xy)^{-1} \text{ for all } x,\ y\in G.$$

Then

$$\alpha'(x,y)^e=\alpha(x,y)^e b(x)^e b(y^e)b(xy)^{-e}$$
$$=a(x)a(y)a(xy)^{-1}\cdot a(x)^{-1}a(y)^{-1}a(xy)=1.$$

Since the number of $|G|^{th}$ roots of 1 are at most $|G|$,

the above proofs say that $|M_G|$ is finite.

It is clear that M_G is a multiplicative group.

Therefore M_G is a finite group.

§ 2 The 2-dimensional Cohomology group of G.

DEFINITION 4. Let A be an abelian group and G a group.

A group extension of A by G is a short exact sequence

$$E:\ o \to A \xrightarrow{k} B \xrightarrow{\sigma} G \to I,$$

Where B and G are not necessary abelian groups and K, σ group homomorphisms and $K(A)$ a normal subgroup of B.

For convenience, we shall write the group composition in B as addition. Let us Put

AutA = the group of automorphisms of A.

Then there is a homomorphism $\theta: B \to \text{AutA}$ which is defined by

$$\begin{array}{ccccc} A & \xrightarrow{\theta(b)} & A & \xrightarrow{K} & B \\ \Cup & & \Cup & & \\ a & \rightsquigarrow & \theta(b)a & \rightsquigarrow & K(\theta(b)a) = b + K(a) - b \end{array}$$

for all $b \in B$. Since B is not abelian $b + K(a) - b \neq b - b + k(a) = 0$ By using this homomorphism θ we can define a homomorphism φ: $G \to \text{AutA}$ with $\theta = \varphi \cdot \sigma$. That is, for $a \in A$ and $b \in B$

$$K[(\varphi[\sigma[b]])a] = b + K(a) - b.$$

Thus, in E A is a G-module, and E is an extension of A by G with operator φ: $G \to \text{Aut } A$. Note that $b + k(a) - b \in K(A)$, simce $K(A)$ is normal in B. Let $AX_\varphi G$, be the semi-direct product of A and G, That is, for (a_1, g_1), $(a_2, g_2) \in A \times G$, We have addition in $A \times_\varphi G$ such that

$$(a_1, g_1) + (a_2, g_2) = (a_1 + \varphi(g_1)a_2, g_1 g_2),$$

So $A \times_\varphi G$ is not abelian, of course

$$\begin{array}{ccccccc} o \to & A & \to & A \times_\varphi G & \to & G & \to I \\ & \Cup & & \Cup \quad \Cup & & \Cup & \\ & a & \rightsquigarrow & (a, I)\ (a, g) & \rightsquigarrow & g & \end{array}$$

is an extension of A by G. We have to note that $A \times G$, $A \times_\varphi G$ and B(in E) are all isomorphic as sets.

DEFINITION 5. Let E: $O \to A \to B \to G \to I$ and E^1: $O \to A \to B_1 \to G \to I$ be two extensions of A by G.

By a morphism $\Gamma: E \to E^1$ we mean a triple $\Gamma = (I_A, \beta, I_G)$ of group homomorphisms such that the diagram

$$
\begin{array}{l}
\qquad\qquad\quad K \qquad \sigma \\
E:\ o \longrightarrow A \longrightarrow B \longrightarrow G \longrightarrow I \\
\qquad\qquad\ \downarrow I_A \quad \downarrow \beta \quad \downarrow I_G \\
E:\ o \longrightarrow A \longrightarrow B_I \longrightarrow G \longrightarrow I
\end{array}
$$

is commutative. By the short five lemma ([2]) β must be an isomorphism.

In this case, $\Gamma=\Gamma(I_A,\beta.I_G)$ is called a congruence, and thus each congruence has inverse. Let us denote the set of all congruence classes of extensions of A by G with operator φ by opext(A,φ,G).

For convenience, Let us put

$$K(a)=a,\ \ \varphi(x)a=xa$$

for $a\in A$ and $x\in G$ in E. For each $x\in G$ we take an element $U(x)$ in $\sigma^{-1}(x)$. Then, from $K[(\varphi\sigma[U(x)])a]=U(x)+K(a)-U(x)$ we have

$$xa+U(x)=a+U(x).$$

On the other hand, for x, $y\in G$ $\sigma[U(X)+U(y)]=xy$ and thus $U(x)+U(y)$ is contained in $\sigma^{-1}(xy)$. Therefore, there exists an element $f_E(x,y)$ of A such that

$$U(x)+U(y)=f_E(x,y)+U(xy).\ \ (*)$$

Since $\sigma(o)=I$, We have $U(I)=0$ and also

$$f_E(X,I)=0=f_E(I,y),\ \ x,y\in G\ \ (*)'$$

Then, fu: $G\times G\to A$ satisfies the following;

(i) For $x,y,z\in G$

$$xf_E(y,z)+f_E(x,yz)=f_E(x,y)+f_E(xy,z)\ \ (**)$$

(ii) If we take an other element $U'(x)$ in $\sigma^{-1}(X)$,
then for $x,y,z\in G$

$$fu'(x,y)=\delta g(x,y)+f_E(x,y),$$

Where

$$(\delta g)(x,y)=xg(y)-g(xy)+g(x),\ \ x,y\in G\ \ (***)$$

and $g:G\to A$ is a function.

(iii) If f: $G\times G\to A$ satisfies$(*)'$ and $(**)$ then f is called a fator set of(A,φ,G). We denote by $\bar{F}$ the set of all factor sets of(A,φ,G). For $f,f'\in\bar{F}$ if there is a function δg: $G\times G\to A$ satisfying$(***)$ such that $f'(x,Y)=\delta g(x,y)+f(x,y)$
then f and f' are said to be isomorphic, written $f\sim f'$. Then$\sim$is an equivalence relation. We put $F=\bar{F}/\sim$

The above function f_E: $G\times G\to A$ in $(*)$ is a factor set of (A,φ,G).

LEMMA 6. As sets, F and Opext (A,φ,G) are isomorphic.

PROOF, For each $\{f\} \in F$ We can Conxtruct an extension

$$\begin{array}{ccccccccc} O & \longrightarrow & A & \longrightarrow & A\times_f G & \longrightarrow & G & \longrightarrow & I \\ & & \Cup & & \Cup & & & & \\ & & a & & (a,I) & \Cup & \Cup & & \\ & & & & & (a,x) & \rightsquigarrow & X & \end{array}$$

of A by G as follows, For $u(x)=(a,x)$, $U(y)=(b,y)(a,b\in A, x, y\in G)$ define the following:

(a) $U(xy)=(a+xb,\ xy)$

(b) $U(x)+U(y)=(a+xb+f[xy], xy)$

Then, by$(*)'$ and$(**)$ We have

$$(O,I)+(a,X)=(a,X)+(o,I)=(a,x)$$
$$[U(x)+U(y)]+U(z)=U(x)+[U(y)+U(z)],$$

and for $b\in A$ with $xb=-a$ we have also

$$(a,x)+(b,x^{-1})=(b,x^{-1})+(a,x)=(o,I)$$
(Note that $-xa=b\Leftrightarrow xb=-a$),

Then $A\times_f G$ is a group AXG with the above abbition(b).

For two factor sets f and $f'\in\{f\}\in F$ we consider two extensions

$$E:\ o\to A\to A\times_f G\to G\to I,$$
$$E':\ o\to A\to A\times_f G\to G\to I$$

of A by G. We define a mapping

$$\begin{array}{cccc} \beta: & AX_f G & \to & AX_f{}'G \\ & \Cup & & \Cup \\ & a+U(x) & \rightsquigarrow & a+g(x)+U'(x) \end{array}$$

Where g: $G\to A$ is a function (Note that each element of $A\times_f G$ can be represented uniquely as $a+U(x)$ for $a\in A$ and $x\in G$).

Since $U(x)+a=xa+U(x)$ We have

$$\beta(a+U(x))+(b+U(y))=a+xb+f(x,y)+g(xy)+U'(xy).$$
$$\beta(a+U(x))+\beta(b+U(y))=a+g(x)+xb+xg(y)+f'(x,y)+U'(xy).$$

By $(**)$

$$\beta(a+u(x))+\beta(b+U(y))=\beta[(a+U(x))+(b+U(y))]$$

and thus β is a homomorphism, Define

$$\begin{array}{ccc} \beta^{-1}: & A\times_{f1}G & \longrightarrow A\times_f G \\ & \Cup & \\ & a+u^1(x) & \rightsquigarrow a-g(x)+U(x), \end{array}$$

then β^1 is a homomorphism and $\beta'\beta=I_A\times_f G$. Therefore, β is an isomorphism and E' is in the congruence class belonging to E'. Similary, We can prove that if Γ: $E\to E'$ is a congruence then $f_E\sim f_E{}'$, where f_E is the factor set of E and $f_E{}'$ the factor set of E'. In consequence, we proved that F and opext (A,φ,G) are isomorphic as sets.

DEFINITION 7. For $\{f\}$, $\{f'\}\in F$ we define

$$\{f\}+\{f'\}=\{f+f'\},$$

where $(f+f')(x,Y)=f(x,y)+f'(x,y)$ for all$(x,y)\in G\times G$.

Then F becomes an abelian group. Since $F\cong\text{Opext}(Z,\varphi,G)$ (as sets), by the abelian group F we can introduce on $\text{Opext}(A,\varphi,G)$ the structure of abelian group, Therefore, Lemma 6 says that $F\cong\text{Opext}(A,\varphi,G)$ as abelian groups, we put

$$\text{OPext}(A,\varphi,G)=H_\varphi{}^2(G,A),$$

which is called the 2-dimensional cohomology group of G with respect to A and φ.

An extension of C^* by G is a short exact sequence

$$E:\ I\to C^*\to B\to G\to I.$$

where B is a multiplicative group and the action of G on C^* is trivial,

i.e, $\forall\gamma\in C^*$ and $\forall x\in G$ $x\gamma=\gamma$. We put

$$\begin{array}{lccc} IC^*: & G & \longrightarrow & \text{Aut } C^* \\ & \Downarrow & & \Downarrow \\ & X & \rightsquigarrow & I\ C^* \end{array}$$

and also Opext $(C^*,I_G,G)=H^2(G,C^*)$

THEOREM 8. $M_G\cong H^2(G,C^*)$

PROOF. In Definition 5, We see that $M_G\cong F$ Using multiplication instead of addition (refer DefInitionI), When

$$\begin{array}{lcccccccc} E: & 1 & \longrightarrow & C^* & \longrightarrow & B & \longrightarrow & G & \longrightarrow o \\ & & & \| & & \cong\downarrow\beta & & \| & \\ E': & 1 & \longrightarrow & C^* & \longrightarrow & B' & \longrightarrow & G & \longrightarrow o \end{array}$$

$\Gamma=T(IC^*,\beta,I_G)$: $E\to E'$ is a congruence. of course

Opext $(C^*,I_G,G)=$the set of all congruence classes of extensions Of C^* by G with operator I_G.

By Lemma 6, as abelian groups $F\cong\text{Opext}(C^*,I,G)=H^2(G,C^*)$.

Thus, We proved that $M_G\cong H^2(G,C^*)$.

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