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A NOTE ON DISTRIBUTIONS ON A RIEMANNIAN SPACE

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ABSTRACT

本 論文의 目的은 리이만 空間 위에 分布論을 導入한 것이다. 따라서 §2가 主內容이 되며 平行分布를 갖는 리이만 空間에서의 局所的 性質을 論한 定理 9가 本論文의 主定理이다.

§1에서는 리이만 距離와 리이만 連結에 대한 定義와 그들간의 關係를 論하였다.

Introduction

The purpose of this note is to introduce the distribution theory on Riemannian spaces. Accordingly, our main part is §2, and our main theorem is theorem 9 which is described a locally property of a Riemannian space with a parallel distribution.

§1 is devoted for §2. In detail, we shall, in §1, describe definitions of Riemannian metric and Riemannian Connection, and moreover, prove a property (proposition 4) which is a relation between Riemannian metric and Riemannian Connection.

§1 Riemannian Spaces

Let M be a n-dimensional differentiable manifold which is paracompact. We put such that

(i) $T_P(M)=$ the tangent space of M at $P(\in M)$.

(ii) $\mathfrak{X}=$ the vector space consisting of all Vector fields on M which are of C^∞-class.

Since M is paracompact, M always has a Riemannian metric g.

In general, (M,g) is called a Riemannian space (or Riemannian manifold). g is a tensor field with degree $(0,2)$ and it has the following properties.

(i) $\forall X, Y \in \mathfrak{X}$

$$g(X,Y)=g(Y,X),\ g(X,Y)\geq 0,\ g(X,X)=0 \rightleftarrows X=0$$

(ii) For a coordinate neighborhood $(U;\ x^1,\ x^2,\cdots\cdots,\ x^n)$ of M,

if $p\in U\left\{\left(\frac{\partial}{\partial x^n}\right)_p,\cdots\cdots,\left(\frac{\partial}{\partial x^n}\right)_p\right\}$ is a base of $T_p(M)$ and if we put $g_{ij} = g_p\left(\left(\frac{\partial}{\partial x^i}\right)_p, \left(\frac{\partial}{\partial x^i}\right)_p\right)$, then the following hold:

$$g_{ij} = g_{ji},\ g_{ji}X^jX^i \geq 0,\ g_{ji}X^jX^i = 0 \Longleftrightarrow X^h = 0,$$

where

$$X = \sum_{i=1}^{n} X^i(p)\left(\frac{\partial}{\partial x^i}\right)_p \in \mathfrak{X}$$

and $X^i: M^n \longrightarrow R$(reals) is a function of c-class.

g_{ji} is called a *component of* g.

DEFINITION 1. A mapping $V: \mathfrak{X} X \mathfrak{X} \rightarrow \mathfrak{X}$ is called an affine connection if it satisfies the conditions (i) ~ (iv):

(i) If $V_x(Y) = V(X,Y)$ then $V_x(Y+Z) = V_x(Y) + V_x(Z)$ for all $X,Y,Z \in \mathfrak{X}$.

(ii) Let us put $\mathfrak{T}$ = the set of all functions from M^n to R which are of c^∞-class. Then, for $f \in \mathfrak{T}$ and $X,Y \in \mathfrak{X}$

$V_x(fY) = fV_x(Y) + (Xf)Y$.

(iii) For

$X,\ Y,\ Z \in \mathfrak{X}$

$V_{x+y}(Z) = V_x(Z) + V_Y(Z)$

(iv) For $X \in \mathfrak{X}$ and $f \in Z$

$V_{fx}(Y) = fV_x(Y)$.

For an affine connection V, if follows from our definition that

(i) $X=0 \Rightarrow V_x(Y)=0,\quad Y=0 \Rightarrow V_x(Y)=0$

(ii) $V_{ax}(Y) = aV_x(Y),\ V_x(aY) = aV_x(Y)\ (a \in R,\ X,Y \in \mathfrak{X})$.

Let us put

$$e_1 = \frac{\partial}{\partial x^1},\ \cdots\cdots,\ e_n = \frac{\partial}{\partial x^n},\ Ve_j(e_i) = \sum_{h=1}^{n} \Gamma^h_{ji}\, e_h = \Gamma^h_{ji}\, e_h,$$

where Γ^h_{ji} is a function of c^∞-class.

Γ^h_{ji} is sometimes called a coefficient of connection V (or connection coefficient).

PROPOSITION 2. With the above notations we have

$$V_x(Y) = X^j\left(\frac{\partial y^h}{\partial x^j} + \Gamma^h_{ji}\, y^i\right)\frac{\partial}{\partial x^h}\ \left(= \sum_{i,j,h} X^j\left(\frac{\partial y^h}{\partial x^j} + \Gamma^h_{ji}\, Y^i\right)\frac{\partial}{\partial x^h}\right)$$

where $X, Y \in \mathfrak{X}$ and

$$X = \sum_{i=1}^{n} X^i e_i, \quad Y = \sum_{j=1}^{n} Y^j e_j.$$

PROOF. $\nabla_X(Y) = \nabla_{x^1 e_1 + \cdots\cdots + x^n e_n} (Y^1 e_1 + \cdots\cdots + Y^n e_n)$

$= \nabla_{x_i e_i} (Y^j e_j)$

$= X^i (\nabla e_i (Y^j e_j))$ (by (iv))

$= X^i (Y^j \nabla e_i (e_j) + e_i (Y^j) e_j)$ (by (iii))

$= X^i (Y^j \Gamma^h_{ij} e_h + \frac{\partial y^h}{\partial x^i} e_h) = X^i (\frac{\partial y^h}{\partial x^i} + Y^j \Gamma^h_{ij}) e_h$

$= X^i \left(\frac{\partial y^h}{\partial x^i} + \Gamma^h_{ij} Y^j \right) \frac{\partial}{\partial x^h}.$

DEFINITION 3. An affine connection ∇ is called a *Riemannian connection* if it satisfied the following conditions:

(i) ∇ is symmetric, *i.e*, for $X, Y \in \mathfrak{X}$

$$[X, Y] = \nabla_X(Y) - \nabla_Y(X),$$

where $[X, Y]$ is the bracket product of X and Y.

(ii) $\nabla_g = 0$, That is

$$\frac{\partial g_{ji}}{\partial x^h} - \Gamma^m_{kj} g_{mi} - \Gamma^m_{ki} g_{jm} = 0,$$

or $\nabla_Z g(X, Y) = g(\nabla_Z(X), Y) + g(X, \nabla_Z(Y))$ for all $X, Y, Z \in \mathfrak{X}$.

We put for components g_{ji} of a Riemannian metric g such that $(g^{ji}) = (g_{ji})^{-1}$, where (g_{ji}) is a $n \times n$ matrix consisting of g_{ji}.

g^{ji} is called a contravarient component of g.

We define $\left\{ {h \atop ji} \right\} = \frac{1}{2} g^{hk} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^k} \right)$,

which is called a Christoffel's symbol.

PROPOSITION 4. For a Riemannian metric g, its component g_{ji} is constant if and only if $\left\{ {h \atop ji} \right\} = 0$.

PROOF. Let ∇ be a Riemannian connection.

We substitude $X = \frac{\partial}{\partial x^j}$, $Y = \frac{\partial}{\partial x^i}$ in the identity

$$[X,Y] = \nabla_X(Y) - \nabla_Y(X), \text{ then}$$
$$\nabla e_j(e_i) - \nabla e_i(e_j) = [e_j, e_i] = 0.$$

Therefore $\Gamma^h_{ji} = \Gamma^h_{ij}$. If we put $\Gamma_{ji,k} = \Gamma^m_{ji} g_{mh}$,

it is obvious that

$$\Gamma_{ji,k} = \Gamma_{ij,h}, \quad \Gamma_{kj,i} + \Gamma_{ki,j} = \frac{\partial g_{ji}}{\partial x^k}.$$

similarly, we get the following;

$$\Gamma_{ik,j} + \Gamma_{ij,k} = \frac{\partial g_{kj}}{\partial x^i}, \quad \Gamma_{ji,k} + \Gamma_{jk,i} = \frac{\partial g_{ik}}{\partial x^j},$$

and thus we have

$$\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ji}}{\partial x^k} = \Gamma_{ik,j} + \Gamma_{ij,k} + \Gamma_{ji,k} + \Gamma_{hk,i} - \Gamma_{kj,i} - \Gamma_{ki,j} = 2\,\Gamma_{ij,k}$$

$$\therefore \Gamma_{ij,k} = \Gamma_{ji,k} = \frac{1}{2}\left(\frac{\partial g_{kj}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ji}}{\partial x^k}\right)$$

since $g_{ji}g^{ki} = \delta_{jk}$ (the kronecker's delta), if we multiply g^{hk} to both sides in the above expression

$$\Gamma^h_{ji} = \frac{1}{2} g^{hk}\left(\frac{\partial g_{ki}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ji}}{\partial x^k}\right) = \left\{ {h \atop ji} \right\}.$$

Therefore,

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{kj,i} + \Gamma_{ki,j} = \Gamma^m_{kj} g_{mi} + \Gamma^m_{ki} g_{mj} = \left\{ {m \atop kj} \right\} g_{mi} + \left\{ {m \atop ki} \right\} g_{mj}$$

Thus, we have

$$g_{ji} = \text{constant} \leftrightarrow \left\{ {h \atop ji} \right\} = 0.$$

§ 2. Distributions on a Riemannian space

DEFINITION 5. In a Remannian space (M, g) a mapping

$$\begin{array}{l} D: M \longrightarrow T(M) = \bigcup_{p \in M} T_p(M) \\ \quad\; \Downarrow \\ \quad\; p \rightsquigarrow D(p) = D_p \subset T_p(M) \end{array}$$

is called a r-dimensional distriibution if for all $p \in M$

D_p is a r-dimensional subspace of $T_p(M)$.

If there are vector fields $X_{(1)},\cdots\cdots, X_{(r)}$ such that

$\{X_{(1)}(p),\cdots\cdots, X_{(r)}(p)\}$ is a base of D_p for all $p\in M$,

then $\{X_{(1)},\cdots\cdots, X_{(r)}\}$ is called a *locally base* of D.

Moreover, there is a submanifold N of M such that for each point $p\in N$ $D_p=T_p(N)$, D is said to be *integrable* and N is called the *integral manifold* of D.

DEFINITION 6. Let D be a r-dimensional distribution of (M,g).

If for every vector fields X and Y in D (*i.e.*, $\forall$ $p\in M$, X_p and Y_p are in D) $[X,Y]_p\in D_p$ for all $p\in M$,

then D is called an *involutive distribution*.

Let ∇ be a Riemannian connection. For $X\in\mathfrak{X}$ and

Y in D(*i.e.*, $\forall$ $Y_p\in D_p$) if $\nabla_X(Y)_p\in D_p$ for all $p\in M$, then D is said to be *parallel*.

It is obvious that a distribution D with dimension r is involutive if and only if for each locally base $X_{(1)},\cdots\cdots, X_{(r)}$

$$[X_{(\alpha)}, X_{(\beta)}]=\sum_{r=1}^{r} F^r_{\alpha\beta}\, x_{(r)} \ (\alpha,\beta=1,2,\cdots\cdots, r),$$

where $F^r_{\alpha\beta}$ is a function of c^∞-class in a coordinate neighborhood. As is well known, a distribution is integrable if and only if it is involutive. Furthermore, if a distribution D is integrable, then the following holds.

In a sufficiently small coordinate neighborhood $(U:x^h)$, the integral submanifold of D is determined by the equations

$$f^{r+1}(x^1,\cdots\cdots, x^n)=c^{r+1},\cdots\cdots, f^n(x^1,\cdots\cdots, x^n)=C^n,$$

where $C^{r+1},\cdots\cdots, C^n$ are constants and $f^{r+1},\cdots\cdots, f^n$ are functions of c^∞-class.

That is, we have

$$x^{r+1}=C^{r+1},\cdots\cdots, x^n=C^n.$$

PROPOSITION 7. A parallel distribution D on a Riemannian space is integrable

PROOF. Since D is parallel, for vector fields X and Y in D

$$\nabla_X(Y),\ \nabla_Y(X)\in D,$$

Where ∇ is a Riemannian connection. Thus

$$[X,Y]=\nabla_X(Y)-\nabla_Y(X)\in D.$$

We put for a distribution D with dimension r such that

$$D_p{}^{\perp} = \{A \in T_p(M) \mid g_p(A,B) = 0 \text{ for all } B \in D_p\},$$

which is called the orthogonal complement space of D_p.
We define a distribution D such that.

$$D^{\perp}: P \rightsquigarrow D_p{}^{\perp}$$

(Note: dim $D_p = r \Rightarrow$ dim $D_p{}^{\perp} = n-r$)

PRPPOSITION 8. If a distribution D is parallel, then so is $D^{\perp}$.

PROOF. For $X \in D$ and $Y \in D^{\perp}$ $g_p(X,Y) = 0$. therefore, for a vector field Z and a Riemannian comnection ∇

$$\nabla_Z g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) = 0$$

since $\nabla_Z X \in D$ $g(\nabla_Z X, Y) = 0$. It follows that $g(X, \nabla_Z Y) = 0$ since $X \in D$ and $g(X, \nabla_Z Y) = 0$ $\nabla_Z Y \in D^{\perp}$, and thus $D^{\perp}$ is parallel.

THFOREM 9. If a Riemannian space (M,g) has a parallel distribution D with dimension r, then (M,g) is locally a product space of a r-dimensional Riemannian space and $(n-r)$-dimensional Riemannian space.

PROOF. since D and $D^{\perp}$ are parallel (by proposition 8) they have integral submanifolds with equations:

$$D: x^{r+1} = C^{r+1}, \cdots\cdots, x^n = C^n$$
$$D^{\perp}: x^1 = C^1, \cdots\cdots, x^r = C^r \ (C^1, \cdots\cdots, C^r: \text{constants}).$$

in a sufficiently small neighborhood $(U; x^h)$.
In particular, in U the components of a Riemannian metric g can be written such that

$$(g_{ji}) = \begin{pmatrix} g_{\beta\alpha} & 0 \\ 0 & g_{\mu\lambda} \end{pmatrix} \qquad (※)$$

$(\alpha, \beta = 1, \cdots\cdots, r;\ \lambda, \mu, = r+1, \cdots\cdots, n)$.

PROOF Let us take $\left\{ \frac{\partial}{\partial x^1} = e_1, \cdots\cdots, \frac{\partial}{\partial x^r} = e_r \right\}$ as a locally base of D (we may do this because of that D is parallel).
Then

$$\nabla e_\beta(e_\alpha) = \left\{ \begin{matrix} h \\ \beta\alpha \end{matrix} \right\} e_h \text{ Note: } \left\{ \begin{matrix} h \\ \beta\alpha \end{matrix} \right\} = \Gamma^h_{\beta\alpha}$$

$(\alpha, \beta, = 1, \cdots\cdots, r)$ is included in D

Therefore

$$\sum_{\lambda=r+1}^{n} \left\{ \begin{matrix} \lambda \\ \beta\alpha \end{matrix} \right\} e_\lambda = 0 \ (\beta, \alpha = 1, 2, \cdots\cdots, r) \qquad (※※)$$

By (∴), (∴∴) and $\frac{\partial g_{\beta\alpha}}{\partial x^\lambda} - \left\{ {m \atop \lambda\beta} \right\} g_{m\alpha} - \left\{ {m \atop \lambda\alpha} \right\} g_{\beta m} = 0$

We get $\frac{\partial g_{\beta\alpha}}{\partial x^\lambda} = 0$ for $1 \leq \alpha, \beta \leq r$ and $r+1 \leq \lambda \leq n$.

(Note: For $1 \leq \alpha \leq r$ and $r+1 \leq \lambda \leq n$

$$0 = \left\{ \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\lambda} = [e_\alpha, e_\lambda] = Ve_\alpha(e_\lambda) - ve_\lambda(e_\alpha), \right.$$

and thus $Ve_\alpha(e_\lambda)$ $(\in D)$ $= Ve_\lambda(e_\alpha)$ $(\in D^1)$. This means that

$$\left\{ {h \atop \lambda\alpha} \right\} e^h = 0, \ i.e, \ \left\{ {h \atop \lambda\alpha} \right\} = \left\{ {h \atop \alpha\lambda} \right\} = 0 \cdot)$$

similarly $\frac{\partial g_{\beta\alpha}}{\partial x^\lambda} = 0$ for $1 \leq \lambda \leq r$ and $r+1 \leq \alpha, \beta \leq n$.

They imply that

(i) $g_{\alpha\beta}$ $(1 \leq \alpha, \ \beta \leq r)$ does not depend on $x^{r+1}, \cdots\cdots,$ and x^n,

(ii) $g_{\lambda r}$ $(r+1 \leq \lambda, \ r \leq n)$ does not depend on $x^1, \cdots\cdots,$ and x^r.

Take a point $P_o \in U$ with coordinate $(x^1_o, \cdots\cdots, x^n_o)$, and put

$$N = \{Q(x^1, \cdots\cdots, x^n) \in M | \ |x^h - x^h{}_o| < d, \ d\text{: positive}\}.$$

Then N is an open set containing P_o of M. Next, in R^r

we put

$$R = (x^1, \cdots\cdots, x^r) | \ |x^h - x^h{}_o| < d \text{ for } h = 1, 2, \cdots\cdots, r\}.$$

and in R^{n-r} we put

$$S = \{(x^{r+1}, \cdots\cdots, x^n | \ |x^h - x^h{}_o| < d \text{ for } h = r+1, \cdots\cdots, n\}.$$

We make Riemmannian metrices g_1 andg_2 on R and S,

respectively such that

(i) the components of g_1 are $g_{\alpha\beta}$ $(x^1, \cdots\cdots, x^r)$ $(1 \leq \alpha, \beta \leq r)$

(ii) the components of g_2 are $g_{\alpha\beta}$ $(x^{r+1}, \cdots\cdots, x^n)$ $(r+1 \leq \alpha, \ \beta \leq n)$.

Then (R, g_1) and (S, g_2) are Riemannian spaces.

Accordingly, it followes that

$$(N, g) = (R, g_1) \times (S, g_2).$$

References

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