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ON THE DIRECT PRODUCTS AND SUMS OF PRESHEAVES

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ABSTRACT

Abelian群의 presheaf에 관한 直積과 直和를 Category 입장에서 定義하고 presheaf F_λ $(\lambda\in\Lambda)$들의 두 直積(또는 直和)은 서로 同型的 關係에 있으며, 특히 ϕ:X→Y가 homeomorphism이라 하고 ϕ_*F를 X上의 presheaf F의 direct image이라 하면

(1) $(\phi_*F,\ \phi_*(f_\lambda)_{\lambda\in\Lambda})$가 $(\phi_*F_\lambda)_{\lambda\in\Lambda}$의 直積일 때 오직 그때 한하여 $(F,\ (f_\lambda)_{\lambda\in\Lambda})$는 $(F_\lambda)_{\lambda\in\Lambda}$의 直積이다.

(2) $(\phi_*F,\ \phi_*(\iota_\lambda)_{\lambda\in\Lambda})$가 $(\phi_*F_\lambda)_{\lambda\in\Lambda}$의 直和일 때 오직 그때 한하여 $(F,\ (\iota_\lambda)_{\lambda\in\Lambda})$는 $(F_\lambda)_{\lambda\in\Lambda}$의 直和이다.

Let $(F_\lambda)_{\lambda\in\Lambda}$ be an indexed set of presheaves of abelian group on topological space X. We can define the cartesian product $\prod_{\lambda\in\Lambda} F_\lambda$ of $(F_\lambda)_{\lambda\in\Lambda}$ by

$$(\prod_{\lambda\in\Lambda} F_\lambda)(U)=\prod_{\lambda\in\Lambda}(F_\lambda(U)) \text{ for U open in X}$$

$$\rho_V^U:\ (\prod_{\lambda\in\Lambda} F_\lambda)(U)\longrightarrow(\prod_{\lambda\in\Lambda} F_\lambda)(V)\ ((s_\lambda)_{\lambda\in\Lambda}\dashrightarrow({}_\lambda\rho_V^U(s_\lambda))_{\lambda\in\Lambda})$$

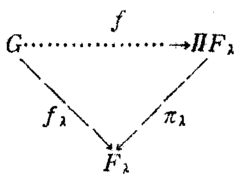
for $V\subseteq U$ open in X where ${}_\lambda\rho_V^U$ is a restriction of F_λ. And we have natural presheaf morphisms π_λ and ι_λ such that

$$\pi_\lambda(U):\ (\prod F_\lambda)(U)\longrightarrow F_\lambda(U)\ ((s_\lambda)_{\lambda\in\Lambda}\dashrightarrow s_\lambda)$$

$$\iota_\lambda(U):\ F_\lambda(U)\longrightarrow(\prod F_\lambda)(U)\ (s_\lambda\dashrightarrow(o,o,\cdots\cdots o,s_\lambda,o,\cdots\cdots o))$$

for $(s_\lambda)\in\prod_\lambda F_\lambda(U)$ and $s_\lambda\in F_\lambda(U)$

PROPOSITION 1. let $(F_\lambda)_{\lambda\in\Lambda}$ be an indexed set of presheaves of abelian group on topological space X. Let G be a presheaf of abelian group on X and $(f_\lambda)_{\lambda\in\Lambda}$ be an indexed set of presheaf morphisms $f_\lambda: G\to F_\lambda$. Then there exists a unique presheaf morphism $f: G\to\prod F_\lambda$ such that for each $\lambda\in\Lambda$, the following diagram



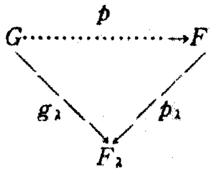
commutes.

PROOF. For each $s\epsilon G(U)$, define $F(U)$: $G(U)\longrightarrow(\Pi F_\lambda)(U)$ by $\pi_\lambda(U)f(U)(s)=f_\lambda(U)(s)$, $(\lambda\epsilon\Lambda)$. Since $\pi_\lambda(U)$ and $f_\lambda(U)$ are homomorphism $f(U):s\dashrightarrow f(U)(s)$ defines a homomorphism $G(U)\longrightarrow(\Pi F_\lambda)(U)$, Suppose that $g:G\longrightarrow\Pi F_\lambda$ is another presheaf morphism such that $\pi_\lambda g=f$ for all $\lambda\epsilon\Lambda$. Then for each $s\epsilon G(U)$ and $\lambda\epsilon\Lambda$, we have $\pi_\lambda(U)g(U)(s)=\pi_\lambda(U)(s)$ so $g(U)(s)=f(U)(s)$. Thus $g=f$.

DEFINITION. Let $(F_\lambda)_{\lambda\epsilon\Lambda}$ be an indexed set of presheaves of abelian group on topological space X. A pair $(F,(P_\lambda)_{\lambda\epsilon\Lambda})$ consisting of a presheaf F on X and presheaf morphism P_λ: $F\longrightarrow F_\lambda$ is called a direct product of $(F_\lambda)_{\lambda\epsilon\Lambda}$, if there exists unique presheaf morphism $f:G\longrightarrow F$ such that $f_\lambda=p_\lambda f$ for each preshaf G on X and each set of presheaf morphisms $f_\lambda:G\longrightarrow F_\lambda$.

From poroposition 1, $(\Pi F_\lambda,(\pi_\lambda)_{\lambda\epsilon\Lambda})$ is a direct product of $(F_\lambda)_{\lambda\epsilon\Lambda}$.

THEOREM 1. Let $(F_\lambda)_{\lambda\epsilon\Lambda}$ be an indexed set of presheaves of abelian group on topological space X. Let$(F,(p_\lambda)_{\lambda\epsilon\Lambda})$ be a direct product of $(F_\lambda)_{\lambda\epsilon\Lambda}$. Then a pair $(G,(q_\lambda)_{\lambda\epsilon\Lambda})$, where each $q_\lambda:G\longrightarrow F_\lambda$ is a presheaf morphism, is alse a direct product of $(F_\lambda)_{\lambda\epsilon\Lambda}$ iff there exists a (necessarily unigue) isomorphism $p:G\longrightarrow F$ such that for each $\lambda\epsilon\Lambda$, the following diagram



commutes.

PROOF. Since $(F,(p_\lambda)_{\lambda\epsilon\Lambda})$ is a direct product, there is a unique presheaf morphism $p:G\longrightarrow F$ with $p_\lambda(U)p(U)=q_\lambda(U)$ for U open in X and each $\lambda\epsilon\Lambda$.

$(\Rightarrow)$ If$(G,q_\lambda)_{\lambda\epsilon\Lambda})$ is direct proiuct of $(F_\lambda)_{\lambda\epsilon\Lambda}$, then there is a unique presheaf morphism $p':F\longrightarrow G$ whth $q_\lambda(U)p'(U)=p_\lambda(U)$ for each $\lambda\epsilon\Lambda$. Then $q_\lambda(U)=p_\lambda(U)p(U)=q_\lambda p'(U)p(U)$. By uniqueness $p'(U)p(U)=I_{G(U)}$. Similarly, $p(U)p'(U)=I_{F(U)}$.

$(\Leftarrow)$ If $f_\lambda:H\longrightarrow F_\lambda,(\lambda\epsilon\Lambda)$. are presheaf morphims, then there exists a unique presheaf morphism $h:H\longrightarrow F$ with $p_\lambda(U)h(U)=f_\lambda(U)$ for each $\lambda\epsilon\Lambda$ since $(F,(p_\lambda)_{\lambda\epsilon\Lambda})$ is a direct product. Since p is an isomorphism, there exists a presheaf morphism $r:G\longrightarrow F$ with $p(U)r(U)=I_{G(U)}$ and $r(U)p(U)=I_{F(U)}$. Let us take

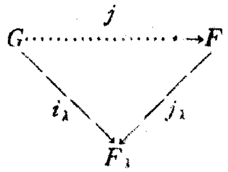
$$f(U)=r(U)h(U):H(U)\longrightarrow G(U).$$

Then $f:H\longrightarrow G$ is a presheaf morphism and $q_\lambda(U)f(U)=q_\lambda(U)r(U)h(U)=p_\lambda(U)h(U)=f_\lambda(U)$. So that $(G,(q_\lambda)_{\lambda\epsilon\Lambda})$ is a direet product.

DEFINITION. Let $(F_\lambda)_{\lambda\epsilon\Lambda}$ be an indexed set of presheaves of abelian group on topological space X. A pair $(F, (j_\lambda)_{\lambda\epsilon\Lambda})$ consisting of a presheaf F on X and presheaf morphism $j_\lambda : F_\lambda \longrightarrow F$ $(\lambda\epsilon\Lambda)$ is called a direct sum of $(F_\lambda)_{\lambda\epsilon\Lambda}$ if there exists a unique presheaf morphism $f : F \longrightarrow G$ such that $f_\lambda(U)=f(U)j_\lambda(U)$ for each presheaf G on X and each set of presheaf morphism $f_\lambda : F_\lambda \longrightarrow G$.

Following theorem is the dual of therem 1

THEOREM 2. theorem 1, Let $(F_\lambda)_{\lambda\epsilon\Lambda}$ be an indexed set of presheaves of abelian group on topological space X. Let $(F\,(j_\lambda)_{\lambda\epsilon\Lambda})$ be a direct sum of $(F_\lambda)_{\lambda\epsilon\Lambda}$. Then a pair $(G, (i_\lambda)_{\lambda\epsilon\Lambda})$, where $i_\lambda : F_\lambda \longrightarrow G$ $(\lambda\epsilon_\Lambda)$, is a presheaf morphism, is also direct sum of $(F_\lambda)_{\lambda\epsilon\Lambda}$ iff there exists a (necessarily unique) isomorphism $j : F \longrightarrow G$ such that for each $\lambda\epsilon\Lambda$ the following diagram



commutes.

Let $(F_\lambda)_{\lambda\epsilon\Lambda}$ be an indexed set of presheaves of abilian group on topological space X. We can define a presheaf $\oplus F_\lambda$ by

$$(\bigoplus_{\lambda\epsilon\Lambda} F_\lambda)\,(U)=\bigoplus_{\lambda\epsilon\Lambda}(F_\lambda(U) \text{ for } U \text{ open in } X.$$

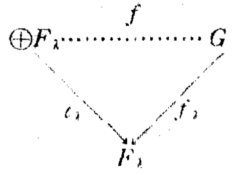
$$\rho^U_V : (\bigoplus_{\lambda\epsilon\Lambda} F_\lambda)\,(U) \longrightarrow (\bigoplus_{\lambda\epsilon\Lambda} F_\lambda)\,(V)\ (s_\lambda)_{\lambda\epsilon\Lambda} \dashrightarrow ({}_\lambda\rho^U_V(s)_{\lambda\epsilon\Lambda})$$

for $V \subseteq U$ open in X. where ${}_\lambda\rho^U_V$ is a restriction of F_λ. We have presheaf morphism ι_λ and π_λ such that

$$\iota_\lambda(U) : F_\lambda(U) \longrightarrow (\oplus F_\lambda)\,(U)\ \ (s_\lambda \dashrightarrow (o, o, \cdots\cdots o, s_\lambda, o, \cdots\cdots o)$$

$$\pi_\lambda(U) : (\oplus F_\lambda)\,(U) \longrightarrow F_\lambda(U)\ \ ((s_\lambda)_{\lambda\epsilon\Lambda} \dashrightarrow s_\lambda)$$

PROPOSITION 2. Let $(F_\lambda)_{\lambda\epsilon\Lambda}$ be an indexed set of presheaves of abelian group on topological space X. Let G be a presheaf abelian groupon X and $(f_\lambda)_{\lambda\epsilon\Lambda}$ be an indexed set of presheaf morphism $f_\lambda : F_\lambda \to G$ for $\lambda\epsilon\Lambda$. Then there exists a unique presheaf morphism f: $\oplus F_\lambda \to G$ such that the following diagram



commutes fore each $\lambda\epsilon\Lambda$.

PROOF For each $s\in\oplus F_\lambda(U)$, let $A(s)=\{\lambda\in\Lambda \mid \pi_\lambda(U)(s)=O\}$. Define $f:\oplus F_\lambda \longrightarrow G$ by $f(U)(s)=\sum_{\lambda\in A(s)} f_\lambda(U)\ \pi_\lambda(U)(s)$. Then f is a unique presheaf morphism such that $f\iota_\lambda=f$. From proposition 2, $(\oplus F_\lambda,\ (\iota_\lambda)_{\lambda\in\Lambda})$ is a direct sum of $(F_\lambda)_{\lambda\in\Lambda}$. Let $(F^\gamma)_{\lambda\in\Lambda}$ be an indexed set of presheaves on topological space X and $(G_\lambda)_{\lambda\in\Lambda}$ be second indexed set of presheaves on X. Let $f_\lambda : F_\lambda \longrightarrow G_\lambda$ be presheaf morphism for each $\lambda\in\Lambda$. If $(F,(p_\lambda)_{\lambda\in\Lambda})$ and $(G, (p_\lambda')_{\lambda\in\Lambda})$ are direct product of $(F_\lambda)_{\lambda\in\Lambda}$ and $(G_\lambda)_{\lambda\in\Lambda}$ respectively, then it is easy to see that there exists a unique mopphisms $f : F \longrightarrow G$ such that $p_\lambda'\ f=f_\lambda p_\lambda(\lambda\in\Lambda)$

$$\begin{array}{ccc} F & \overset{f}{\cdots\cdots\longrightarrow} & G \\ {\scriptstyle p_\lambda}\downarrow & & \downarrow{\scriptstyle p_\lambda'} \\ F_\lambda & \overset{f_\lambda}{\longrightarrow} & G_\lambda \end{array}$$

Dually, if $(H,(j_\lambda)_{\lambda\in\Lambda})$ and $(K,(j_\lambda)_{\lambda\in\Lambda})$ are direct sum of $(F_\lambda)_{\lambda\in\Lambda}$ and $(G_\lambda)_{\lambda\in\Lambda}$ respectively, then there exists a unique morphism $h : H\rightarrow K$ such that $h\ j_\lambda=j_\lambda' f_\lambda\ (\lambda\in\Lambda)$.

$$\begin{array}{ccc} H & \overset{h}{\cdots\cdots\longrightarrow} & K \\ {\scriptstyle j_\lambda}\uparrow & & \uparrow{\scriptstyle j_\lambda'} \\ F_\lambda & \overset{f_\lambda}{\longrightarrow} & G_\lambda \end{array}$$

Particulary, in case of direct product $(\Pi F_\lambda,\ (\pi_\lambda)_{\lambda\in\Lambda})$ and $(\Pi G_\lambda,\ (\pi_\lambda')_{\lambda\in\Lambda})$ the unique morphism $f : \Pi F_\lambda \longrightarrow \Pi G_\lambda$ is

$$f(U) : \Pi F_\lambda(U) \longrightarrow \Pi G_\lambda(U)\ \ ((x_\lambda)_{\lambda\in\Lambda} \dashrightarrow (f_\lambda(U)(x_\lambda)_{\lambda\in\Lambda})$$

So that $\text{Im}\, f(U)=\Pi\ \text{Im}\, f_\lambda(U)$, $\text{Ker}\, f(U)=\Pi\ \text{Ker}\, f_\lambda(U)$. From $\text{Ker}\, f(U)=\Pi\ \text{Ker}\, f_\lambda(U) =(\Pi\ \text{Ker}\, f_\lambda)(U)$, kernal of f is a product of kernel of f_λ.

Suppose we are given a continuous map $\varphi : X\rightarrow Y$ of topological spaces and a presheaf F on X, we obtain the direct image of F by φ (deneted by $\varphi_* F$).Given a continuous map φ, φ_* is a functor.

Let F and G be presheaves on X. $\varphi_*(F\times G)(U)=(F\times G)(\varphi^{-1}(U))=F(\varphi^{-1}(U)\times G(\varphi^{-1}(U))=(\varphi_* F)(U)\times(\varphi_* G)(U)=(\varphi_* F\times\varphi_* G)(U)$ for U open in Y and $\varphi_*(F\times G)(i)=(\varphi_* F\times\varphi_* G)(i)$ for $i : V \longrightarrow U$. Therefore $\varphi_*(F\times G)=\varphi_* F\times\varphi_*\ G$.

THEOREM 3. Let $\varphi:X \longrightarrow Y$ be a homeomorphism. Then

(1) $(F,\ (f_\lambda)_{\lambda\in\Lambda})$ is a direct product of $(F_\lambda)_{\lambda\in\Lambda}$ iff $(\varphi_*(F),\ \varphi_*(f_\lambda)_{\lambda\in\Lambda})$ is a direct product of $(\varphi_* F_\lambda)_{\lambda\in\Lambda}$.

(2) $(F,\ (\iota_\lambda)_{\lambda\in\Lambda}$ is adirect sum of $(F_\lambda)_{\lambda\in\Lambda}$ iff $(\varphi_* F,\ \varphi_*(\iota_\lambda)_{\lambda\in\Lambda})$ is a direct sum of $(\varphi_* F_\lambda)_{\lambda\in\Lambda}$.

PROOF. We shall do (1) because the proof of (2) is dual. Since $\varphi:X\longrightarrow Y$ is a homeomorphism. there exist a continuous map $\Psi:Y\longrightarrow X$ such that $\Psi\varphi=1_X$, $\varphi\Psi=1_Y$. The Ψ_* is functor such that $\Psi_*\phi_*=1$, $\phi_*\Psi_*=1$.

($\Rightarrow$). Suppose that $(F,(f_\lambda)_{\lambda\in\Lambda})$ is a direct product of $(F_\lambda)_{\lambda\in\Lambda}$. For each morphism $h_\lambda:H\longrightarrow\Phi_*F_\lambda$, there exists $\Psi_*(h_\lambda):\Psi_*H\longrightarrow F_\lambda$ and so there exists a unique morphism $f:\Psi_*H\longrightarrow F$ such that $\Psi_*(h_\lambda)=f_\lambda f$ for $\lambda\in\Lambda$.

This $\phi_*(f)$ is such that $h_\lambda=\phi_*(f_\lambda)\ \phi_*(f)$ and is unique morphism $H\longrightarrow\phi_*\ F$ such that $h_\lambda=\phi_*(f_\lambda)\ \phi_*(f)$ for; If another morphism $h:H\rightarrow\phi F$ such that $h_\lambda=\phi_*(f_\lambda)\ h$, then

$$\Psi_*(h_\lambda)=f_\lambda\Psi_*(h)=f_\lambda f.$$

From uniqueness of f, $\Psi_*(h)=f$.

Therefore $h=\phi_*(f)$.

($\Leftarrow$). Suppose that $(\phi_*F,\ \phi_*(f_\lambda)_{\lambda\in\Lambda})$ is a direct product of $(\phi_*F_\lambda)_{\lambda\in\Lambda}$. For each morphism $k_\lambda:K\rightarrow F_\lambda$, there exists a morphism $\phi_*(k_\lambda):\phi_*K\longrightarrow\phi_*F_\lambda$ and there exists a unique morphism $k:\varphi_*K\longrightarrow\phi_*F$ such that $\phi_*(k_\lambda)=\phi_*(f_\lambda)\ k$,

Therefore, there exists a unique morphism $\Psi_*(k):K\longrightarrow F$ such that $k_\lambda=f_\lambda\Psi_*(k)$ for each morphism $k_\lambda:K\longrightarrow F_\lambda$.

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