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A NOTE ON SMOOTH AFFINE VARIETIES

KWANG-HO, SO

ABSTRACT

本 論文에서는 smooth Affine variety가 微分可能 多樣體임을 보임으로써 代數幾何學이 多樣體理論과 關聯됨을 論하였다.

§*1* 에서는 affine variety의 次元과 affine variety의 接空間에 對한 定義와 그에 關聯된 性質들을 論하였고

§*2* 에서는 simple point와 局所 媒介變數, 그리고 θ_x에서의 有理函數의 Taylor 級數 等을 利用하여 主定理(定理9)를 證明하였다.

Introduction

In this note, we shall prove that a smooth affine variety is a differentiable manifold (Theorem 9).

This shows that algebraic geometry is connected with theory of manifolds.

In §*1*, we shall define the following:

(i) dimensions of the affine variety

(ii) tangent space of the affine variety at a point in the variety.

Also, we shall prove some propesties about tangent space (Proposition 3 and 4).

In §*2*, we shall define the following:

(i) simple points

(ii) local parameters

(iii) Taylor series of a rational function in θ_x.

The above things ie deseribed to prove Theorem 9.

In order to make this note we refer the books [1] and [2].

§*1*. Tangent spaces

Let k be an algebraically closed field, and let us put $T=(T_1,\cdots\cdots, T_n)$. For an algebraic variety X is given by a system of equations

$$F_1(T)=0,\cdots\cdots,\ F_m(T)=0,$$

we put such that

A^n = the n-dimensional affine space over k.

$k[X]$ = the coordinate ring of X.

$\mathfrak{a}_x = (F_1, \cdots\cdots, F_m) \subset k[T]$.

Then, $X \subset A^n$. As is well known, for an irreducible subvariety Y of X, $k[Y]$ is a prime ideal of $k[x]$.

Similary, for each $x \in X$ there exists a prime ideal $P = M_x = \{f \in k[X] \mid f(x) = 0\}$, $k[X]_P = \theta_x$ is the local ring of the point x. For a point $x \in X$ we assume that the system of coordinates in A^n is chosen so that $x = (0, \cdots\cdots, 0) = O$. A line $L \subset A^n$ passing through O is $L = \{ta \mid t \in k\}$, where $a = (a_1, \cdots\cdots, a_n)$ is a fixed point other than O.

DEFINITION 1 The intersection multiplicity at a point O of a line L and an algebraic variety X is the multiplicity of the root $t = 0$ in the polynomial $f(t) = g.c.d.\ (F_1(ta)$ $\cdots\cdots, F_m(ta)$, where $g.c.d.$ = greatest common divisor.

If the polynomials $F_i(ta)$ vanish identically, then the intersection multiplicity is taken to be $+\infty$.

DEFINITION 2 A line L touches the variety X at O if their intersection multiplicity at this point is greater than 1. The locus of points on lines touching X at $x (\in X)$ is called the tangent space at the point x, which is denoted by Θ_x (or $\Theta_{x},\ x$).

PROPOSITION 3 In the same notations as above, the equations of the tangent space Θ_o are

$$L_1(a) = \cdots\cdots = L_m(a) = 0,$$

where $L_i (i = 1, \cdots\cdots, m)$ is the linear part of F_i.

PROOF For $i = 1, \cdots\cdots, m$, since $F_i(0) = 0$ the free terms of all the polynomials $F_i(T)$ are zero, we put for $i = 1, \cdots\cdots, m$ such that $F_i = L_i + G_i$,
where L_i is the linear part of F_i and G_i consists of terms of degree at least 2 in F_i. In this case it follows that $F_i(at) = tL_i(a) + G_i(at)$.

Since the inteisection multiplicity of $L = \{ta \mid t \in k\}$ and X is at least 2, $F_i(at)$ must be divided by t^2. This equivalent to the equations $L_1(a) = \cdots\cdots = L_m(a) = 0$.

For $F(T) \in k[X]$, we can define a regular function $F: X \to k$ such that for each $x \in X$ $F(x)$. For each $x \in X$, let F_1 be the linear part of F and we denote this by d_xF. Then, for $F(T)$ and $G(T)$ in $k[X]$ it follows that.

(i) $d_x(F+G) = d_xF + d_xG$

(ii) $d_x(F.G) = d_xF \cdot G(x) + F(x) \cdot d_xG$

Let Θ_x^* be the dual space of Θ_x.

PROPOSITION 4 $M_x/(M_x)^2$ is ismorphic to Θ_x^* under the homomorphism d_x.

PROOF Recall that $M_x=\{f\in k[X] \mid f(x)=0\}$. Thus Im $(d_x: M_x\to\Theta_x^*)=\Theta_x^*$ is clear. We assume that $d_x g=0$, where $g\in M_x$. Since $\mathfrak{A}_x=(F_1,\cdots\cdots,F_m)$ and $d_x g=0$ on Θ_x^* by our assumption, $d_x g$ is a linear combination of $d_xF_1,\cdots\cdots$, and d_xF_m, that is,

$$d_x g=\lambda_1 d_xF_1+\cdots\cdots+\lambda_m d_xF_m \quad (\lambda_i\epsilon k).$$

We put $G_1=g-(\lambda_1F_1+\cdots\cdots+\lambda_mF_m)$. Then G_1 does not contain any terms of degree 0 or 1 in $T_1,\cdots\cdots, T_u$.

In this case $G_1 \mid X=g$ and $g\epsilon(t_1,\cdots\cdots,tn)^2$ where $t_i=T_i \mid X$ for $i=1,\cdots\cdots,n$ (Note that $(t_1,\cdots,t_n)=m_x$ in this case).

As we well know, the dimension of tangent space at $x\epsilon X$ is invariant under some isomorphisms.

§2. Simple points

With the notations in §1, $k(X)$ is the quotient field of the integral domain $k(X)$. Then, the dimension of X (dim X) is the transcendence degree of $k(X)$ over k.

DEFINITION 5 Let X be an affine variety. A point $x\epsilon X$ is said to be simple if dim Θ_x=dim X, where dim Θ_x is the dimension of the vector space Θ_x over k. A variety X is called a smooth variety if all its points are simple.

A point $x\epsilon X$ is said to be singular if it is not simple.

DEFINITION 6 Functions $u_1,\cdots\cdots, u_n\epsilon\theta_x$ are called local parameters at $x\epsilon X$ if $u_i\epsilon M_x$ and $u_1,\cdots\cdots,u_n$ form a base of the space $M_x/(M_x)^2$.

By the isomorphism d_x: $M_x\to M_x/(m_x)^2$, $u_1,\cdots\cdots$, u_n form a system of local parameters if and only if the linear forms $d_xu_1,\cdots\cdots,d_xu_n$ are linear independent on Θ_x. Since dim $\Theta_x=n$, the conditions of local parameters is equivalent to the fact that the equations

$$d_xu_1=\cdots\cdots d_xu_n=0$$

have only the trivial solution in Θ_x.

PROPOSITION 7 The system of local parameters generate the maximal ideal M_x of the local ring θ_x.

PROOF We have seen that $M_x=(t_1,\cdots\cdots,t_N)$ where $X\subset A^N$ and $x=(0,\cdots\cdots,0)\epsilon A^N$. If we assume that $u_1,\cdots\cdots,u_n$ are local parameters at x, then we have to claim that $t_i\epsilon(u_1,\cdots\cdots,u_n)$ $(i=1,\cdots\cdots,N)$. At first, by induction on N-i we shall prove that $t_i\epsilon$ $(u_1,\cdots\cdots,u_n, t_1,\cdots\cdots,t_{i-1})$.

Suppose that $M_x=(u_1,\cdots\cdots,u_n,\ t_1,\cdots\cdots,\ t_N)=(u_1,\cdots\cdots,\ u_n,\ t_1,\cdots\cdots,\ t_i)$.

Then, since $u_1,\cdots\cdots,\ u_n$ are base of $M_x/(m_x)^2$

$$t_i=\sum_{j=1}^{n} a_j u_j(m_x^2),\quad a_j\epsilon k.$$

But, every element of M_x^2 is of the form

$$\mu_1\ u_1+\cdots\cdots+\mu_n u_n+\mu_1' t_1+\cdots\cdots+\mu_i' t_i\ \ (\mu_j\ \mu_j'\ \epsilon m_x),$$

and thus

$$t_i=\sum_{j=1}^{n} a_j\ u_j+\sum_{j=1}^{n}\mu_j\ u_j+\sum_{j=1}^{i}\mu_j' t_j.$$

Therefore we have

$$(1-\mu_i')t_i=\sum_{j=1}^{n} a_j\ u_j+\sum_{j=1}^{i-1}\mu_j' t_j\ \epsilon(u_1,\cdots\cdots,\ u_n,t_1,\cdots\cdots,\ t_{i-1}).$$

Since $\mu_i'\epsilon m_x$, $1-\mu_i\notin m_x$, and thus $(1-\mu_i')^{-1}\in\theta_x$.

Therefore, in Θ_x $t_i\epsilon(u_1,\cdots\cdots,\ u_n,\ t_1,\cdots\cdots,\ t_{i-1})$. It is easily prove that $t_N\epsilon(u_1,\cdots\cdots,u_n,$ $t_1,\cdots\cdots,\ t_{N-1})$.

It is clear that x is a simple point if and only if dim $X=$dim (m_x/m_x^2). In general, the dimension of a variety X at x can be defined as the smallest integer γ for which there is γ functions $u_1,\cdots\cdots,\ u_\gamma\epsilon m_x$ such that the set determined by the equations $u_1=\cdots=u_\gamma=0$ consists in a neighborhood of x. In this case $(u_1,\cdots\cdots,\ u_\gamma)\supset M_x^l$ for some $l>0$ by Hilbert's Nullstellensatz.

Let $k[[T]]$ be the ring of formal power series in variables $T=(T_1,\cdots\cdots,\ T_n)$.

Then each $\phi\epsilon k[[T]]$ can be represented by

$$\phi = F_0 + F_1 + F_2 +\cdots\cdots,$$

where $F_i\epsilon k[T]$ is a form of degree i,

DEFINITION 8 A formal power series ϕ is called a Taylor series of the function $f\epsilon\theta_x$ if for all $l\geqq 0$

$$f-Se(u_1,\cdots\cdots,\ u_n)\ \epsilon M_x^{l+1},\ Se=\sum_{i=0}^{l} F_i.$$

Then, if x is a simple point there exists only one Taylor series of the function $f\epsilon\theta_x$. Therefore, there is a monomorphism

$$\tau:\ \theta_x \rightarrow k[[T]],$$

that is the mapping τ is an isomorphic embedding of the local ring θ_x in the ring of formal power series $k[[T]]$.

Furthermore, by using this statements we can prove that there is only one component of X papping through a simple point x, where a component is an irreducible subvariety of X containing x.

THFOREM 9 Let k be the field of real or complex numbers and let X be a smooth affine variety. Then the formal Taylor series of functions $f \in \theta_x$ converge for small values $T_1, \cdots\cdots, T_n$.

Therefore X is a differentiable manifold.

PROOF Let $\mathfrak{a}_x = (F_1, \cdots\cdots, F_m)$, $X \subset A^N$ and dim $X = n$. If $x \in X$ is a simple point of X then dim $\Theta_x = n$ and Θ_x is determined by the equations

$$d_x F_1 = \sum_{i=1}^{N} \left(\frac{\partial F_1}{\partial T_i}\right)(x)(T_i - x_i) = 0, \cdots\cdots, \quad d_x F_m = \sum_{i=1}^{N} \left(\frac{\partial F_m}{\partial T_i}\right)(x)(T_i - x_i) = 0$$

Therefore the rank of matrix

$$\left(\frac{\partial F_i}{\partial T_j}(x)\right) \; (i = 1, \cdots\cdots, m, \; j = 1, \cdots\cdots, N)$$

is N-n. Thus we may assume that

$$\left| \frac{\partial F_i}{\partial T_j}(x) \right| \neq 0 \; (i = 1, \cdots, N\text{-}n, \; j = n+1, \cdots, n+m) \cdots\cdots(1)$$

Let x be the origin of coordinates. Then $t_1, \cdots\cdots, t_n$ form a system of local parameters of x. Let X' be the set of components passing through x of the variety which is determined by the equations

$$F_1 = 0, \cdots\cdots, F_{N-n} = 0 \; (N - n \leqq m). \cdots\cdots\cdots\cdots\cdots\cdots(2)$$

Then, we can prove that $X' = X$. So, we may take X as neighborhood of x by the equations of (2) with (1) holding.

By the implicit function theorem there exists a system of power series $\phi_1, \cdots\cdots, \phi_{N-n}$ in the n-variables $T_1, \cdots\cdots, T_n$ and $\epsilon > 0$ such that $\phi_j(T_1, \cdots\cdots, T_n)$ converges for all T_i with $|T_i| < \varepsilon$ and

$$F_i(T_1, \cdots\cdots, T_n, \phi_1(T), \cdots\cdots, \phi_{N-n}(T)) = 0 \cdots\cdots\cdots\cdots(3)$$

where the coefficients of $\phi_1, \cdots\cdots, \phi_{N-n}$ are uniquely determined and $i = 1, \cdots\cdots, N-n$.

By (2) the formal power series $\tau(t_{n+1}), \cdots\cdots, \tau(t_N)$ ($t_1, \cdots\cdots, t_n$ are a system of local parameters) satisfy (3). Therefore $j=1,\cdots\cdots,N-n$ $\phi_j=\tau(t_{n+j})$ and $\tau(t_{n+j})$ converges for $|T_i|<\varepsilon$ where $i=1,\cdots\cdots,n$.

For each function $f\epsilon\theta_x$

$$f=p(t_1,\cdots\cdots,\ t_N)/Q(t_1,\cdots\cdots,\ t_N)\ Q(x)\neq 0$$

and
$$\tau(f)=p(\tau(t_1),\cdots\cdots,\ \tau(t_N))/Q(\tau(t_1),\cdots\cdots,\ \tau(t_N)).$$

Therefore the series $\tau(f)$ converge for $|T_j|<\varepsilon$ $j=1,\cdots\cdots,$ n. In general, the series $\tau(f)$ has a positive radius of convergence for any choice of local parameters. Thus, the inplicit function theorem says the following.

(i) there are convergent power series $\phi_1,\cdots\cdots,$ ϕ_{N-n}

(ii) for some η and for any point $(t_1,\cdots\cdots,\ t_N)\epsilon X$ with $|t_i|<\eta,$ $t_{n+i}=\phi_i(t_1,\cdots\cdots,t_n)$ $(i=1,\cdots\cdots,\ N-n)$ (see (3))

It follows that the mapping

$$(t_1,\cdots,\ t_N)\longmapsto(t_1,\cdots\cdots,\ t_n)$$

carries the set $(t_1,\cdots\cdots,\ t_N)\ \epsilon X,$ $|t_i|<\eta$ $(i=1,\cdots\cdots,\ n)$, one-to-one and bicontinuously onto a domain of the n-dimensional space.

(Note: $(t_1,\cdots\cdots,\ t_N)=(t_1,\cdots\cdots,\ t_n,\ \phi_1(t_1,\cdots\cdots,\ t_n)\cdots\cdots,\ \phi_{N-n}(t_1,\cdots\cdots,\ t_n))$ and $F(t_1,\cdots\cdots,\ t_N)=0$).

Therefore, each point $x\epsilon X$ has an open neighborhood which is homeomorphic to an open subset in k^n.

That is, X is a differentiable manifold.

REFERENCES

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Jeonbug National University.