HONAM MATHEMATICAL JOURNAL
Volume 1, Number 1, July 1979

ON THE INTEGRAL THEORY OVER DIFFERENTIABLE MANIFOLDS (I)

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ABSTRACT

Positive Local Coordinate$(x^1, x^2, \cdots x^n)$을 갖는 Oriented Manifold M을 생각한다.

M이 Compact Carrier를 갖는 경우, M위의 n次 Differential Form $\psi^{(n)}$의 積分을

$$\int \psi^{(n)} = \sum_{\alpha} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_\alpha \psi^{(n)} dx^1 \cdots dx^n$$

로 定義하며(定義 7),

M위의 p 次의 Differential form $\beta^{(p)}$와 Differential simplex $S^{(p)}=(S^{(p)}, \pi, \varepsilon)$에 대하여 $S^{(p)}$위의 $\beta^{(p)}$의 積分을

$$\int_{^{(p)}S} \beta^{(p)} = \int_{S^{(p)}} \varepsilon\pi^* \beta^{(p)} = \int_{E^p} f.\varepsilon.\pi^* \beta^{(p)}$$

로 定義한다(定義 9).

但 $\bar{S}^{(p)}$는 $S^{(p)}=(p_0 \cdot p_1 \cdots p_p)$에 依하여 Spanning 되는 E^p의 Subspace이고 f는 $\bar{S}^{(p)}$의 特性函數이다.

이때 $(n-1)$次의 Differential Form $\beta^{(n-1)}$이 Compact인 Carrier를 가지면

$\int d\beta^{(n-1)}=0$ 이 됨을 考察하며(定理 8),

$(p-1)$次 Differential Form $\beta^{(p-1)}$과 p次 Differential Chain $C^{(p)}$에 關하여

$$\int_{C^{(p)}} d\beta^{(p-1)} = \int_{\partial C^{(p)}} \beta^{(p-1)}$$

이 成立함을 究明하려 한다(定理 10).

§ 1. Introduction

The purpose of this paper is to introduce the integral theory over differertiable manifolds(Definition 7 and Definition 9) and to prove two Theorems(Theorem 8 and Theorem 10).

To dothese we describe definitions in section 2 about differentiable forms(Definition 1) and differentiable chains(Definition 6).

§ 2. Differentiable forms and Differentiable chains

Let M be *a* n-dimensional differentiable manifold over R(reals) with system of coordinate neighborhoods $\{(U_\alpha, \phi_\alpha)\}\alpha\epsilon A$.

We assume the following:

(i) M is of C^∞-class and paracompact,

(ii) for each $p\epsilon M$ $T_p(M)$ is the tangent space of M at p and $T(M)=\bigcup_{p\epsilon M} T_p(M)$, thus $\{T(M), \pi, M\}$ is the tangent bundle.

(iii) X: $M\to T(M)$ is a cross section of C^∞-class. In particular, for $p\epsilon M$ with local coordinate$(x^1,\cdots,x^n)$

$$X(p)=\sum_{i=1}^{n}X^i(p)(\frac{\partial}{\partial x^i})(p)$$

(iv) $\mathfrak{X}$ is the set of all cross sections of C^∞-class, then $\mathfrak{X}$ is a vector space over R.

(v) $\mathfrak{X}^*$ is the dual space of $\mathfrak{X}$ and $\{(dx^1)_p,\cdots,(dx^n)_p\}$ is the dual base of $\left\{(\frac{\partial}{\partial x^1})_p,\cdots\right.$ $\left.\cdots,(\frac{\partial}{\partial x^n})_p\right\}$ at $p\epsilon M$.

DEFINITION 1. A map $\varphi:M\to\overset{p}{\wedge}\mathfrak{X}^*$($p$-fold exterior power of $\mathfrak{X}^*$) is called a p-degree differential form if for each $Q\epsilon M$ and its local coordinate$(x^1,\cdots,x^n)$

$$\varphi(p)=\sum_{i_1<\cdots<i_p}\varphi_{i_1\cdots i_p}(Q)(dx^{i_1})_Q\cdots(dx^{i_p})_Q,$$

where $\varphi_{i_1\cdots i_p}$: $M\to R$ is a function of C^∞-class. We denote φ by

$$\varphi=\sum_{i_1<\cdots<i_p}\varphi_{i_1\cdots i_p}dx^{i_1}\cdots dx^{i_p}$$

All p-degree differential forms over M becomes a R-module under usual addition and scalar product.

DEFINITION 2. A differential operator "d" is a linear operator from the R-module of p-degree differential forms satisfying the conditions:

(i) $d(\varphi\pm\psi)=d\varphi\pm d\psi$,

(ii) $d(\phi^{(p)}, \psi^{(q)})=d\varphi^{(p)}\cdot\psi^{(q)}+(-1)^p\phi^{(p)}d\psi^{(q)}$,

(iii) $d(d\varphi)=0$,

(iv) if f: $M\to R$ is of C^∞-class, ther df coincides with the differential of f, where $0\leq p,q\leq n$, $\varphi^{(p)}$ is a p-degree differential form and φ is a differential form.

PROPOSITION 3. For each $\varphi=\sum_{i_1<\cdots<i_p}\varphi_{i_1\cdots i_p}dx^{i_1}\cdots dx^{i_p}$, if we define

$$d\varphi=\sum_{i_0}\sum_{i_1<\cdots<i_p}\frac{\partial\varphi_{i_1\cdots i_p}}{\partial x^{i_0}}dx^{i_0}dx^{i_1}\cdots dx^{i_p},$$

then "d" is a differential operator.

PROOF Conditions(i) and(iv) are obvious.

For a differential form

$$\psi=\sum_{j_1<\cdots<j_q}\phi_{j_1\cdots j_q}\,dx^{j_1}dx^{j_q},$$

we have

$$\varphi\phi=\sum_{\substack{i_1<\cdots<i_p\\ j_1<\cdots<j_q}}\varphi_{i_1\cdots i_p}\,\phi_{j_1\cdots j_q}\,dx^{i_1}\cdots dx^{i_p}\cdot dx^{j_1}\cdots dx^{j_q}$$

In this case, we have

$$d(\varphi\psi)=\sum_k\sum_{i,j}\left(\frac{\partial\varphi_{i_1\cdots i_p}}{\partial x^k}\psi_{j_1\cdots j_q}+\varphi_{i_1\cdots i_p}\frac{\partial\psi_{j_1\cdots j_q}}{\partial x^k}\right)\cdot dx^k$$

$$dx^{i_1}\cdots dx^{i_p}dx^{j_1}\cdots dx^{j_q}$$

$$=\left\{\sum_k\sum_i\frac{\partial\varphi_{i_1\cdots i_p}}{\partial x^k}dx^k\,dx^{i_1}\cdots dx^{i_p}\right\}\left\{\sum_j\phi_{j_1\cdots j_q}\,dx^{j_1}\cdots dx^{j_q}\right\}$$

$$+\left\{\sum_i\varphi_{i_1\cdots i_p}dx^{i_1}\cdots dx^{i_p}\right\}\left\{(-1)^p\sum_k\sum_j\frac{\partial\phi_{j_1\cdots j_q}}{\partial x^k}dx^k\,dx^{j_1}\cdots dx^{j_q}\right\}$$

$$=d\varphi.\psi+(-1)^p\,\varphi.d\psi,$$

and then (ii) is proved.

Next, we also have

$$d(d\varphi)=\sum_{k_1<k_2}\sum_{i_1<\cdots<i_p}\left(\frac{\partial^2\varphi}{\partial x^{k_1}\partial x^{k_2}}-\frac{\partial^2\varphi}{\partial x^{k_2}\partial x^{k_1}}\right)dx^{k_1}dx^{k_2}dx^{i_1}\cdots dx^{i_p}=0,$$

which means that (iii) is true.

DEFINITION 4 (Orientation of Manifolds)

Let M be a manifold as above, and let π_p and π_p' be n-fold differential forms over R at: $p\in M$ $(\pi_p\neq 0=\pi_p')$. If $a>0$, then we assume that π_p and π_p' belong to the same class, and if $a<0$ then we assume that π_p and π_p' belong to the different class each other. In these two classes we take one class K_p, and we call $(T_p(M),\ K_p)$ an oriented vector space, and $\pi_p\ \epsilon K_p$ a positive n-fold differential form.

For the base $\left\{\left(\frac{\partial}{\partial x^1}\right)_p,\cdots\left(\frac{\partial}{\partial x^n}\right)_p\right\}$ of $T_p(M)$, if $\pi_p\left\{\left(\frac{\partial}{\partial x^1}\right)_p, \cdots \left(\frac{\partial}{\partial x^n}\right)_p\right\} > 0$, then

$\left\{\left(\frac{\partial}{\partial x^1}\right)_p, \cdots \left(\frac{\partial}{\partial x^n}\right)_p\right\}$ is called a positive base.

Consider $\{(T_p(M),\ K_p)\}_{P\epsilon M}$ and a n-fold differential form φ. For each point p and its open neighborhood $U(p)$, if $\varphi\ \epsilon K_p$ and for each point $q\epsilon U_p$ we have $\varphi\ \epsilon K_q$, then M is said to be orientable and the mapping $p(\epsilon M) \longrightarrow (T_p(M),\ K_p)$ is called an orientation of M. Let $p \longrightarrow (T_p(M),\ K_p)$ be an orientation, and let $(x^1,\ x^2, \cdots x^n)$ be a local coordinate of $p\epsilon M$. If $dx^1, \cdots, dx^n\ \epsilon K_p$, then $(x^1, x^2, \cdots, x^n)$is called a positive local coordinate of p.

PROPOSITIOR 5. If $(x^1,\ x^2, \cdots x^n)$ is a positive local coordinate at $p\epsilon M$, then it is positive on a coordinate neighborhood $U\alpha$ of p.

PROOF. In $U\alpha$, we define such that

$E^+ = \{Q\epsilon U\alpha\colon\ (x^1 \cdots x^n)$ is positive on $Q\epsilon U\alpha, \}$,

then E^+ is an open subset of M because of that if $Q\ \epsilon E^+$, then there exists an open set $U(Q) \subset E^+$ with $Q\epsilon U(Q)$.

Similary, if we define

$E^- = \{Q\epsilon U\alpha\colon\ (x^1 \cdots x^n)$ is negative on $Q\epsilon U\alpha, \}$,

then E^- is an open set.

Since $E^+ \cup E^- = U\alpha$, $E^+ \cap E^- = \phi$ and $U\alpha$ is connected we have $E^- = \phi$ and $E^+ = U\alpha$.

DEFINITION 6(Differential chains)

① Let $S^{(p)} = (P_0 \cdots P_p)$ be a p-dimensional standard simplex with orientation ϵ, and let Π: $E^p \rightarrow M$ be a function of C^∞-class, where E^p is the p-dimensional Euclidean space. In this case,

$$\mathrm{S}^p = (S^p,\ \Pi,\ \varepsilon)$$

is called a differentiable simplex of C^∞-class over M with dimension p.

② A linear combination of differentiable simplexes with dimension p;

$$C^{(p)} = \sum k_i\ \mathrm{S}_i^{(p)},\quad k_i \epsilon R,$$

is called a C^∞-differentiable chain over M.

③ For $S^{(p)} = (P_0 \cdots P_p)$ if we define

$$S_i^{(p-1)} = (P_0\ P_1 \cdots P_{i-1}\ P_{i+1} \cdots P_p),\quad [S^{(p)}\colon S_i^{(p-1)}] = (-1)^i,$$

then the boundary of $S^{(p)}$ is

$$\partial S^{(p)} = \sum_{i=0}^{p} (-1)^i S_i^{(p-1)},$$

where $[S^{(p)}: S_i^{(p-1)}]=(-1)^i$ is the incidence number of $S_i^{(p-1)}$ with respect to $S^{(p)}$.

④ For $\mathbf{S}^{(p)}=(S^{(p)}, \Pi, \varepsilon)$ we define

$$\Pi|S_i^{(p-1)}=\Pi_i,$$

then we have

$$\mathbf{S}_i^{(p-1)}=(S_i^{(p-1)}, \Pi_i, \varepsilon)$$

and the boundary of $\mathbf{S}^{(p)}$ is

$$\partial\mathbf{S}^{(p)}=\sum_{i=0}^{p}(-1)^i\,\mathbf{S}_i^{(p-1)}.$$

⑤ Similary the boundary of differentiable chain

$C^{(p)}=\sum k_i\ \mathbf{S}_i^{(p)}$ is $\partial C^{(p)}=\sum k_i\ \partial\ \mathbf{S}_i^{(p)}$.

§ 3. Integral Theory

DEFINITION 7. Recall that our manifold $\{M, (U_\alpha, \varphi_\alpha)_{\alpha\in A}\}$ is paracompact, so that there exists a partition $\{f_\alpha\}_{\alpha\in A}$ of unity with respect to $\{U_\alpha\}_{\alpha\in A}$.
Let M be an oriented manifold, and let $(x^1\cdots x^n)$ be a positive local coordinate of M. For a n-fold differential form $\phi^{(n)}$ over M with compact carrier, we define its integral such that

$$\sum_\alpha\int_{-\infty}^{\infty}\cdots\cdots\int_{-\infty}^{\infty}f_\alpha\phi^{(n)}dx^1\cdots dx^n,$$

written $\int\phi^{(n)}$.

Then $\int\ \phi^{(n)}$is not depend on the way taking local coordinate and partition of unity.

THEOREM 8. Let $\beta^{(n-1)}$ be a differential form with degree$(n-1)$. If its carrier is compact, then

$$\int d\beta^{(n-1)}=0$$

PROOF. By our definition and (ii) of definition 2 we have

$$\int d\beta^{(n-1)}=\sum_\alpha\int f_\alpha d\beta^{(n-1)}=\sum_\alpha\left\{\int d(f_\alpha\beta^{(n-1)})-\int(df_\alpha)\ \beta^{(n-1)}\right\},$$

where $\{f_\alpha\}_{\alpha\in A}$ is a partition of unity with respect to $\{U_\alpha\}_{\alpha\in A}$. Since $\{f_\alpha\}_{\alpha\in A}$ is locally finite,

$$\sum_\alpha\int(df_\alpha)\ \beta^{(n-1)}=\int\sum_\alpha\{(df_\alpha)\cdot\beta^{(n-1)}\}=\int(\sum_\alpha df_\alpha)\cdot\beta^{(n-1)}$$

$$=\int d(\sum_{\alpha} f_{\alpha})\ \beta^{(n-1)}=\int d1\cdot\beta^{(n-1)}=0.$$

Therefore we have

$$\int d\beta^{(n-1)}=\sum_{\alpha}\int d(f_{\alpha})\ \beta^{(n-1)}.$$

Since the carrier of f_{α} is in U_{α}, we may assume that the carrier of $\beta^{(n-1)}$ is also in some coordinate neighborhood U_{α}.

If we define

$$\beta^{(n-1)}=\sum b_{23}\cdots{}^{n}(x)dx^{2}\cdots dx^{n},$$

then the carrier of $b_{23}\cdots{}^{n}(x)$ is in U_{α}.

Since $d\beta^{(n-1)}=\sum\dfrac{\partial b_{23}\cdots{}^{n}}{\partial x^{1}}dx^{1}\cdots dx^{n}$,

we have

$$\int d\beta^{(n-1)}=\int_{-\infty}^{\infty}\cdots\left(\int_{-\infty}^{\infty}\frac{\partial b_{23}\cdots{}^{n}}{\partial x^{1}}\,dx^{1}\right)dx^{2}\cdots dx^{n}.$$

On the other hand,

$$\int_{-\infty}^{\infty}\frac{\partial b_{23}\cdots{}^{n}}{\partial x^{1}}\,dx^{1}=0,$$

and thus we have

$$\int d\beta^{(n-1)}=0.$$

Q. E. D.

DEFIMTION 9. Let $\beta^{(p)}$ be a differential form with degree p, and let $\mathbf{S}^{(p)}=(S^{(p)},\Pi,\varepsilon)$ be a differentiable simplex over M. We define the integral of $\beta^{(p)}$ on $\mathbf{S}^{(p)}$ such that

$$\int_{S^{(p)}}\beta^{(p)}=\int_{\bar{S}^{(p)}}\varepsilon\Pi^{*}\beta^{(p)}=\int_{E^{p}}f\varepsilon\Pi^{*}\beta^{(p)},$$

where $\bar{S}^{(p)}$ is the subspace of E^{p} spanning by $S^{(p)}=(P_{0},\ P_{1}\cdots P_{p})$, and f is the characteristic function of $\bar{S}^{(p)}$.

Furthermore, we may regard as $\Pi^{*}\beta^{(p)}=\beta^{(p)}$. Let $C^{(p)}=\sum_{i=1}^{n}k_{i}S_{i}^{(p)}$ be a differentiable chain. Then the integral of $\beta^{(p)}$ on $C^{(p)}$ is defined by

$$\int_{C^{(p)}}\beta^{(p)}=\sum_{i=1}^{n}k^{i}\int_{S_{i}^{(p)}}\beta^{(p)}$$

THEOREM 10 (Stokes Theorem)

Let $\beta^{(p-1)}$ be a differential form over M with degree$(p-1)$. For a differentiable chain

$$C^{(p)} = \sum_{i=1}^{n} k_i S_i^{(p)}$$

the following holds:

$$\int_{C^{(p)}} d\beta^{(p-1)} = \int_{\partial C^{(p)}} \beta^{(p-1)}$$

PROOF. Without loss of generality, we may prove our theorem with respect to $S^{(p)}=(S^{(p)}, \Pi, \varepsilon)$, and we may assume that $\varepsilon=1$.

For a point Q of $\bar{S}^{(p)}$, we can take $(x^1\cdots x^p)$ as a positive local coordinate of Q,

$$P_0Q = x^1 P_0P_1 + \cdots + x^p P_0P_p,\ 0 \leqq x^i \leqq 1,\ 0 \leqq \sum x^i \leqq 1.$$

Then we can put

$$\Pi^* \beta^{(p-1)} = \sum_{i=1}^{p} b_{12\cdots\hat{i}\cdots p}\, dx^1 \cdots \widehat{dx^i} \cdots dx^p.$$

In this case,

$$\int_{\partial S^{(p)}} \beta^{(p-1)} = \sum_{j=0}^{p} (-1)^j \int_{S_j^{(p-1)}} \left\{ \sum_{i=1}^{p} b_{1\cdots\hat{i}\cdots p}\, dx^1 \cdots \widehat{dx^i} \cdots dx^p \right\}$$

$$= \sum_{i=1}^{p} \sum_{j=0}^{p} (-1)^j \int_{S_i^{(p-1)}} b_{1\cdots\hat{i}\cdots p} \cdots dx^1 \cdots \widehat{dx^i} \cdots dx^p$$

In particular, we have to note that

(i) $x^j=0$ $(j\neq 0)$ on $S_j^{(p-1)}$,

(ii) $\sum_{i=1}^{p} x^i = 1$ on $S_0^{(p-1)}$,

$$i.e.,\ x_i = 1-(x^1+\cdots+\widehat{x^i}+\cdots+x^p).$$

Therefore,

$$\int_{\partial S^{(p)}} \beta^{(p-1)} = \sum_{i=1}^{p} \left\{ \int_{S_0^{(p-1)}} b_{1\cdots\hat{i}\cdots p}(x^1\cdots x^i\cdots x^p)\, dx^1 \cdots \widehat{dx^i} \cdots dx^p \right.$$

$$\left. + (-1)^i \int_{S_i^{(p-1)}} b_{1\cdots\hat{i}\cdots p}(x^1\cdots \overset{(i)}{0} \cdots x^p)\, dx^1 \cdots \widehat{dx^i} \cdots dx^p \right\}.$$

We define a map of C^∞-class by

$$\mu^{-1}:\ S_0^{(p-1)} \longrightarrow (-1)^i\, \mathbf{S}_i^{(p-1)}$$

$$\qquad (x^1\cdots x^i\cdots x^p) \cdots\cdots\cdots \rightarrow (-1)^{i-1}(x^1\cdots\hat{x}^i\cdots x^p),$$

then we can get

$$S_0^{(p-1)}=\{S_i^{(p-1)},\ \mu,\ (-1)^{i-1}\},$$

and thus

$$\int_{S_0^{(p-1)}} b_{1\cdots \hat{i}\cdots p}(x^1\cdots x^i\cdots x^p)dx^1\cdots\widehat{dx^i}\cdots dx^p$$

$$=\int_{S_i^{(p-1)}}(-1)^{i-1}b_{12\cdots \hat{i}\cdots p}(x^1\cdots x^i\cdots x^p)dx^1\cdots\widehat{dx^i}dx^p$$

$$=\int_{S_i^{(p-1)}}(-1)^{i-1}b_{12\cdots \hat{i}\cdots p}\left\{x^1\cdots x^i(x^1,\cdots\widehat{x^i}\cdots x^p),\cdots x^p\right\}dx^1\cdots\widehat{dx^i}\cdots dx^p.$$

Therefore

$$\int_{\partial C^{(p)}}\beta^{(p-1)}=\sum_{i=1}^{p}(-1)^{i-1}\int_{S_i^{(p-1)}}[(b_{1\cdots \hat{i}\cdots p}(x^1\cdots x^i\cdots x^p)$$

$$-b_{1\cdots \hat{i}\cdots p}(x^1\cdots \overset{(i)}{0}\cdots x^p)]dx^1\cdots\widehat{dx^i}\cdots dx^p.$$

On the other hand, since $\Pi^*\beta^{(p-1)}$ is of C^∞-class,

$$d\Pi^*\beta^{(p-1)}=\sum_{i=1}^{p}(-1)^{i-1}\frac{\partial b_{1\cdots \hat{i}\cdots p}}{\partial x^i}dx^1\cdots dx^p,$$

and thus,

$$\int_{S^{(p)}}d\beta^{(p-1)}=\int_{S^{(p)}}d(\Pi^*\beta^{(p-1)})$$

$$=\sum_{i=1}^{p}(-1)^{i-1}\int_{S^{(p)}}\frac{\partial b_{1\cdots \hat{i}\cdots p}}{\partial x^i}dx^1\cdots dx^p$$

$$=\sum_{i=1}^{p}(-1)^{i-1}\int_{S_i^{(p-1)}}\left\{\int_0^{x^i}\frac{\partial b_{1\cdots \hat{i}\cdots p}}{\partial x^i}dx^i\right\}dx^1\cdots\widehat{dx^i}\cdots dx^p$$

$$=\sum_{i=1}^{p}(-1)^{i-1}\int_{S_i^{(p-1)}}\left\{b_{1\cdots \hat{i}\cdots p}(x^1\cdots x^n)-b_{1\cdots \hat{i}\cdots p}(x^1\cdots\overset{(i)}{0}\cdots x^p)\right\}dx^1\cdots\widehat{dx^i}\cdots dx^p.$$

Therefore, we have

$$\int_{\partial S^{(p)}}\beta^{(p-1)}=\int_{S^{(p)}}d\beta^{(p-1)}.$$

Q. E. D.

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