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ON COMPLEX CONFORMAL CONNECTIONS IN AN ALMOST COMPLEX MANIFOLD WITH A TORSION TENSOR

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§ 1. Introduction

Let M be an n-dimensional Riemannian manifold with metric tensor g_{ji}. The change of the metric

$$\bar{g}_{ji}=e^{2p}g_{ji},$$

where p is a certain scalar function, does not change the angle between two vectors at a point and so is called a conformal change of the metric.

If there exists a function p such that the Riemannian manifold with metric tensor $e^{2p}g_{ji}$ is locally Euclidean, the Rimannian manifold is said to be conformally flat.

It is well known (Weyl [1]) that the so-called Weyl conformal curvature tensor

$$W_{kji}{}^{h}=K_{kji}{}^{h}+\delta_k^h C_{ji}-\delta_j^h C_{ki}+C_k{}^h g_{ji}-C_j{}^h g_{ki}$$

is invariant under a conformal change of g, where $K_{kji}{}^{h}$ is the Riemann-Christoffel curvature tensor of M and

$$C_{ji}=-\left(\frac{1}{n-2}\right)K_{ji}+\left(\frac{1}{2(n-1)(n-2)}\right)Kg_{ji},$$

$$C_k{}^h=C_{ks}g^{sh},\quad K_{ji}=K_{sji}{}^{s},\quad K=g^{ji}K_{ji}$$

and a necessary and sufficient condition for M to be conformally flat is that

$$W_{kji}{}^{h}=0 \quad \text{for } n>3$$

and

$$\nabla_k C_{ji}-\nabla_j C_{ki}=0 \text{ for } n=3,$$

∇_k denoting the operator of covariant differentiation with respect to Christoffel symbols formed with g.

A complex analogue of the above in a Kaehler manifold is given by K. Yano (K. Yano [2]).

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In a Kaehler manifold M with the Hermitian metric tensor g_{ji} and complex structure tensor $F_i{}^h$, we have

$$\nabla_k g_{ji}=0,\quad \nabla_k F_i{}^h=0,\quad \nabla_k F_{ji}=0,$$

where $F_{ji}=F_j{}^s g_{si}$ and consequently $F_{ji}=-F_{ij}$.

The affine connection which satisfies

$$D_k e^{2p} g_{ji}=0,\quad D_k e^{2p} F_{ji}=0$$

and torsion tensor

$$1/2(\Gamma_{ji}{}^h-\Gamma_{ij}{}^h)=-E_{ji}q^h,$$

where p is a scalar function and q^h is a vector field, is given by

$$\Gamma_{ji}{}^h=\left\{{h \atop ji}\right\}+p_i\delta_j^h+p_j\delta_i^h-p^h g_{ji}+q_jF_i{}^h+q_iF_j{}^h-q^hF_{ji},$$

where

$$p_i=\partial_i p,\quad p^h=p_s g^{sh},\quad q_i=-p_sF_i{}^s,\quad q^h=q_s g^{sh}.$$

For this connection called a complex conformal connection, K. Yano proved the following theorem:

If in an n-dimensional Kaehler manifold $(n\geq 4)$, there exists a scalar function p such that the complex conformal connection is of zero curvature, then the Bochner curvature tensor of the manifold vanishes.

Define $C_{kji}{}^h$ by

$$\begin{aligned}C_{kji}{}^h=K_{kji}{}^h&+\delta_k^hL_{ji}-\delta_j^hL_{ki}+g_{ji}L_k{}^h-g_{ki}L_j{}^h\\&+F_k{}^hM_{ji}-F_j{}^hM_{ki}+M_k{}^hF_{ji}-M_j{}^hF_{ki}\\&+\left(\frac{2}{n+4}\right)V_{kj}F_i{}^h+F_{kj}B_i{}^h,\end{aligned}$$

where

$$V_{kj}=1/2K_{kjs}{}^tF_t{}^s,$$

$$M_{ji}=-K_{js}F_i{}^s,$$

$$M_{ji}=-\frac{1}{2(n-2)}(H_{ji}+H_{ij})-\frac{1}{n+4}V_{ji}+\frac{1}{n^2-4}\left(\frac{n+1}{n+4}V-1/2K\right)F_{ji},$$

$$L_{ji}=M_{js}F_i{}^s,$$

$$L_k{}^h=L_{ks}g^{sh},\quad M_k{}^h=M_{ks}g^{sh},\quad B_i{}^h=B_{is}g^{sh},$$

$$B_{ji}=H_{ji}+nM_{ji}-2M_{ij}-\frac{1}{n+2}\left(\frac{1}{n+4}V+1/2K\right)F_{ji},$$

$V=V_{st}F^{st}$ and $K=K_{st}g^{st}$.

This tensor is called a complex conformal curvature tensor. The complex conformal curvature tensor $C_{kji}{}^h$ is invariant under a change of the complex conformal connection. (O.K. Yoon [5]).

A complex analogue of the above in an almost complex manifold with a torsion tensor is not yet known. The main purpose of the present paper is to try to find some properties concerning this problem. In § 2 we state some of fundamental formulas in an almost complex manifold with a torsion tensor to fix our notation and in § 3, we introduce what we call complex conformal connections in an almost complex manifold with a torsion tensor.

In § 4, we study the integrability condition for an almost complex manifold M with a torsion tensor, in § 5, we state some of fundamental formulas, and in § 6 we study an invariant curvature tensor.

§ 2. Preliminaries

We consider an n-dimensional almost complex manifold M with a torsion tensor $S_{ji}{}^h=1/2(\Gamma_{ji}{}^h-\Gamma_{ij}{}^h)$ which is covered by a system of coordinate neighborhoods $(U;\xi^h)$ and denote by g_{ji} and $F_i{}^h$ the components of the Hermitian metric tensor and those of the complex structure tensor of M respectively, where here and in the sequel the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, n\}$.

We denote by D_j the operator of covariant differentiation with respect to $\Gamma_{ji}{}^h$, then the torsion tensor $S_{ji}{}^h=1/2(\Gamma_{ji}{}^h-\Gamma_{ij}{}^h)$ is given by

$$S_{ji}{}^h=-u^hF_{ji}, \tag{2.1}$$

where u^h are components of a vector field. Then, we have

$$D_kg_{ji}=0,\quad D_kF_j{}^h=0,\quad D_kF_{ji}=0, \tag{2.2}$$

where $F_{ji}=F_j{}^sg_{si}$ is skew symmetric.

Above all, we notice that an affine connection is symmetric, that is, which satisfies

$$D_kg_{ji}=0 \tag{2.3}$$

and whose torsion tensor

$$1/2(\Gamma_{ji}{}^h-\Gamma_{ij}{}^h)=S_{ji}{}^h \tag{2.4}$$

is uniquely determined and given by

$$\Gamma_{ji}{}^h=\left\{ {h \atop ji} \right\}+S_{ji}{}^h+S_{ji}{}^h+S_{ij}{}^h, \tag{2.5}$$

where (Hayden [6])

$$S_{ji}{}^h=S_{tj}{}^sg^{th}g_{si}. \tag{2.6}$$

So we have, for the components Γ_{ji}^{h} of affine connection in an almost complex manifold M with a torsion tensor $S_{ji}^{h}=-u^{h}F_{ji}$

$$\Gamma_{j}{}^{h}{}_{i}=\left\{{h \atop ji}\right\}+u_{j}F_{i}^{h}+u_{i}F_{j}^{h}-u^{h}F_{ji}, \tag{2.7}$$

where $\left\{{h \atop ji}\right\}$ are the Christoffel symbols formed with g_{ji} and

$$F^{h}{}_{j}=g^{hs}F_{sj}=-F_{js}g^{sh}=-F_{j}{}^{h},\quad u_{i}=u^{s}g_{si}.$$

We denote by

$$K_{kji}{}^{h}=\partial_{k}\left\{{h \atop ji}\right\}-\partial_{j}\left\{{h \atop ki}\right\}+\left\{{h \atop ks}\right\}\left\{{s \atop ji}\right\}-\left\{{h \atop js}\right\}\left\{{s \atop ki}\right\} \tag{2.8}$$

the components of the Riemann-Christoffel curvature tensor of M, where $\partial_{k}=\partial/\partial\xi^{k}$.

It is well known that $K_{kji}{}^{h}$ and $K_{kjih}=K_{kji}{}^{s}g_{sh}$ satisfy

$$K_{kjih}=-K_{kjih},\quad K_{kjih}=-K_{kjhi}, \tag{2.9}$$

$$K_{kjih}=K_{hijk}, \tag{2.10}$$

$$K_{kjih}+K_{jikh}+K_{ikjh}=0 \tag{2.11}$$

and

$$\nabla_{e}K_{kji}{}^{h}+\nabla_{k}K_{jei}{}^{h}+\nabla_{j}K_{eki}{}^{h}=0, \tag{2.12}$$

$$\nabla_{s}K_{kji}{}^{s}=\nabla_{k}K_{ji}-\nabla_{j}K_{ki}, \tag{2.13}$$

$$2\nabla_{s}K_{k}^{s}=\nabla_{k}K, \tag{2.14}$$

where

$$K_{ji}=K_{ij}=K_{sji}{}^{s}\ \text{and}\ K=g^{ji}K_{ji}$$

are the Ricci tensor and the scalar curvature symbol $\left\{{h \atop ji}\right\}$, respectively.

§ 3. Integrability conditions

We consider the integrability conditions in an almost complex manifold M with a torsion tensor.

From (2.7), using $D_{k}F_{ji}=0$, we have

$$D_{k}F_{ji}=\nabla_{k}F_{ji}-u_{i}g_{kj}+u_{j}g_{ki}-w_{j}F_{ki}+w_{i}F_{kj}=0,$$

where $w_{j}=-u_{s}F_{j}{}^{s}$, consequently

$$\nabla_{k}F_{ji}=g_{kj}u_{i}-g_{ki}u_{j}-F_{kj}w_{i}+F_{ki}w_{j}. \tag{3.1}$$

Then, from the Ricci identity, we obtain

$$(3.2)\qquad \nabla_l\nabla_k F_{ji}-\nabla_k\nabla_l F_{ji}=K_{lkjs}F_i^{\ s}-K_{lkis}F_j^{\ s}$$
$$=g_{kj}u_{li}-g_{lj}u_{ki}-g_{ki}u_{lj}+g_{li}u_{kj}$$
$$-F_{kj}w_{li}+F_{lj}w_{ki}+F_{ki}w_{lj}-F_{li}w_{kj},$$

where

$$(3.3)\qquad u_{ji}=\nabla_j u_i-u_jw_i+1/2\rho F_{ji},$$

$$(3.4)\qquad w_{ji}=\nabla_j w_i-w_jw_i+1/2\rho g_{ji}=-u_{ks}F_j^{\ s},$$

$$(3.5)\qquad \rho=u_su_tg^{st}=w_sw_tg^{st}.$$

If we define

$$(3.6)\qquad K_{sjit}F^{st}=A_{ji},$$

$$(3.7)\qquad K_{js}F_i^{\ s}=-H_{ji}$$

then, from (2.9), (2.10) and (2.11), we have

$$(3.8)\qquad K_{stji}F^{st}=K_{jist}F^{st}=-2A_{ji},\quad A_{ji}+A_{ij}=0.$$

And, from (3.2), we have also

$$(3.9)\qquad A_{ji}-H_{ji}=(n-3)u_{ji}+u_{ij}+(u_{st}F^{st})F_{ji},$$

consequently

$$(3.10)\qquad 2A_{ji}-(H_{ji}-H_{ij})=(n-4)(u_{ji}-u_{ij})+2(u_{st}F^{st})F_{ji},$$

$$(3.11)\qquad H_{ji}+H_{ij}=-(n-2)(u_{ji}+u_{ij}),$$

$$(3.12)\qquad A-K=2(n-2)(u_{st}F^{st}),$$

where $A=A_{st}F^{st}$, $K=K_{st}g^{st}=H_{st}F^{st}$.
From (3.10), (3.11) and (3.12), we find

$$(3.13)\qquad u_{ji}=-\frac{1}{2(n-2)}(H_{ji}+H_{ij})$$
$$+\frac{1}{2(n-4)}\left\{2A_{ji}-(H_{ij}-H_{ij})-\frac{A-K}{n-2}F_{ji}\right\}\ \text{for } n>4,$$

$$(3.14)\qquad 2A_{ji}-(H_{ji}-H_{ij})-\frac{A-K}{2}F_{ji}=0\quad \text{for } n=4.$$

On the other hand, since the Nijenhuis tensor

$$N_{ji}^{\ \ h}=F_j^{\ s}(\partial_sF_i^{\ h}-\partial_iF_s^{\ h})-F_i^{\ s}(\partial_sF_j^{\ h}-\partial_jF_s^{\ h})$$

of the complex structure tensor F_i^h vanishes in virtue of (3.1), the complex structure tensor F_i^h be integrable.

Thus, we have

PROPOSITION 3.1. *In an n-dimensional Riemannian manifold $M(n>4$, even number) with metric tensor g_{ji}, the necessary and sufficient condition such that the manifold M can be admissible an almost complex structure F_i^h which satisfies*

$$D_k g_{ji}=0,\quad D_k F_{ji}=0 \text{ and } \Gamma_{ji}^h-\Gamma_{ij}^h=-2F_{ji}u^h$$

is follows;

$$K_{kjih}=-K_{jkih},\quad K_{kjih}=-K_{kjhi},\quad K_{kjih}=K_{hijk},$$

$$K_{kjih}+K_{jikh}+K_{ikjh}=0,$$

$$\nabla_l K_{kji}{}^h+\nabla_k K_{jli}{}^h+\nabla_j K_{lki}{}^h=0$$

and
$$K_{lkjs}F_i{}^s-K_{lkis}K_j{}^s=g_{kj}u_{li}-g_{lj}u_{ki}-g_{ki}u_{lj}+g_{li}u_{kj}$$
$$-F_{kj}w_{li}+F_{lj}w_{ki}+F_{ki}w_{lj}-F_{li}w_{kj},$$

where, $u_{ji}=\dfrac{1}{2(n-2)}(H_{ji}+H_{ij})$

$$+\frac{1}{2(n-4)}\left\{2A_{ji}-(H_{ji}-H_{ij})-\frac{A-K}{n-2}F_{ji}\right\},$$

$$w_{ji}=u_{is}F_i{}^s.$$

If $n=4$, the last condition can be replaced by

$$2A_{ji}-(H_{ji}-H_{ij})-\frac{A-K}{2}F_{ji}=0,$$

where

$$A_{ji}=K_{sjit}F^{st},\quad K_{ji}=K_{sjit}{}^s,\quad H_{ji}=-K_{js}F_i{}^s$$

and

$$A=A_{st}F^{st},\quad K=K_{st}g^{st}.$$

§4. Some formulas in an almost complex manifold M with a torsion tensor

We denote by

(4.1)
$$R_{kji}{}^h=\partial_k\Gamma_{ji}^h-\partial_j\Gamma_{ki}^h+\Gamma_{ks}^h\Gamma_{ji}^s-\Gamma_{js}^h\Gamma_{ki}^s,$$

the components of the curvature tensor of M.

By a straightforward computation, from (2.7), we find

$$(4.2)\quad \begin{aligned} R_{kji}{}^h = K_{kji}{}^h - F_k{}^h u_{ji} + F_j{}^h u_{ki} - u_k{}^h F_{ji} + u_j{}^h F_{ki} \\ + F_i{}^h (v_{kj} - u_{jk} - \rho F_{kj}) - 2F_{kj}(u_i w^h - u^h w_i) \\ + u_k(\delta_j^h u_i - g_{ji} u^h + F_j{}^h w_i - F_{ji} w^h + F_i{}^h w_j) \\ - u_j(\delta_k^h u_i - g_{ki} u^h + F_k{}^h w_i - F_{ki} w^h + F_i{}^h w_k), \end{aligned}$$

where $w^h = w_s g^{sh}$, from which $R_{kjih} = R_{kji}{}^s g_{sh}$ satisfy

$$(4.3)\quad R_{kjih} = -R_{jkih},\quad R_{kjih} = -R_{kjhi},$$

$$(4.4)\quad \begin{aligned} R_{kjih} - R_{hijk} \\ = -F_{kj} X_{(ji)} + F_{jh} X_{(ki)} - F_{ji} X_{(kh)} + F_{ki} X_{(jh)} \\ + F_{ih} X_{[kj]} - F_{kj} X_{[ih]}, \end{aligned}$$

$$(4.5)\quad R_{kjih} + R_{jikh} + R_{ikjh} = -2F_{kj} X_{ih} - 2F_{ji} X_{kh} - 2F_{ik} X_{jh},$$

where $X_{(ji)} = X_{ji} + X_{ij}$, $X_{[ji]} = X_{ji} - X_{ij}$,
and

$$(4.6)\quad X_{ji} = u_{ji} + 2u_j w_i - u_i w_j + (\delta/2) F_{ji}.$$

Using $D_k F_{ji} = 0$, from the Ricci identity, we have

$$(4.7)\quad R_{kjis} F_h{}^s = R_{kjhs} F_i{}^s.$$

Now, if we define

$$(4.8)\quad R_{stji} F^{st} = -2E_{ji},\ R_{jist} F^{st} = -2V_{ji},\quad R_{sjit} F^{st} = A_{ji},$$

then, from (4.4) and (4.5), we have

$$(4.9)\quad 2V_{ji} - 2E_{ji} = -nX_{ji} + 2XF_{ji},$$

$$(4.10)\quad A_{ji} - A_{ij} - 2V_{ji} = 2X_{ji} - 2XF_{ji},$$

$$(4.11)\quad A_{ji} - E_{ji} = -(\mathrm{n}-2) X_{ji},$$

where $X = X_{st} F^{st}$, and consequently if we put

$$A = A_{st} F^{st},\quad E = E_{st} F^{st} (= V_{st} F^{st}),$$

then we have

$$(4.12)\quad X = \frac{1}{n-2}(E - A).$$

From (4.11), we have

$$(4.13)\quad X_{[ji]} = \frac{1}{n-2}\{2E_{ji} - (A_{ji} - A_{ij})\},\quad X_{(ji)} = -\frac{1}{n-2}(A_{ji} + A_{ij}),$$

furthermore, eliminating X_{ji} from (4.9), (4.10) and (4.11) we have

$$(4.14)\qquad n(A_{ji}-A_{ij})-4E_{ji}-2(n-2)V_{ji}+2(E-A)F_{ji}=0.$$

Substituting (4.11) and (4.13) into (4.4) and (4.4), we have

$$(4.15)\qquad R_{kjih}-R_{hijk}=\frac{1}{n-2}\{F_{kj}A_{(ji)}-F_{jh}A_{(ki)}+F_{ji}A_{(kh)}$$
$$-F_{ki}A_{(jh)}-F_{ih}(A_{kj}-2E_{kj})+F_{kj}(A_{ih}-2E_{ih})\},$$

$$(4.16)\qquad R_{kjih}+R_{jikh}+R_{ikjh}$$
$$=\frac{2}{n-2}\{F_{kj}(A_{ih}-E_{ih})+F_{ji}(A_{kh}-E_{kh})+F_{ik}(A_{jh}-E_{jh})\}.$$

On the other hand, from the Bianchi identity, we have

$$(4.17)\qquad D_lR_{kjih}+D_kR_{jlih}+D_jR_{lkih}$$
$$=2u^s(F_{lk}R_{sjih}+F_{kj}R_{slih}+F_{jl}R_{skih}),$$

hence, by contracting with F^{kj}, we obtain

$$(4.18)\qquad u^sR_{slih}=\frac{1}{n-2}(F^{st}D_sR_{tlih}-D_lE_{ih}).$$

Substituting (4.18) into (4.17), we have

$$(4.19)\qquad D_lR_{kjih}+D_kR_{jlih}+D_jR_{lkih}$$
$$=\frac{2}{n-2}\{F_{lk}(F^{st}D_sR_{tjih}-D_jE_{ih})+F_{kj}(F^{st}D_sR_{tlih}-D_lE_{ih})$$
$$+F_{jl}(F^{st}D_sR_{tkih}-D_kE_{ih})\}.$$

Therefore, we have the following

PROPOSITION 4.1. *In an n-dimensional* $(n\geq 4)$ *almost complex manifold M with Hermitian metric tensor* g_{ji}, *complex structure tensor* $F_i^{\ h}$, *the affine connection which satisfies*

$$D_kg_{ji}=0,\quad D_kF_{ji}=0$$

and

$$\Gamma_{ji}^{\ \ h}-\Gamma_{ij}^{\ \ h}=-2F_{ji}u^h,$$

where u^h *is a vector field, is given by*

$$\Gamma_{ji}^{\ \ h}=\left\{ {h \atop j\,i} \right\}+F_j^{\ h}u_i+F_i^{\ h}u_j-F_{ji}u^h.$$

Furthermore, for the curvature tensor, the following relations hold;

$$R_{kjih}=-R_{jkih},\quad R_{kjih}=-R_{kjhi},\quad R_{kjis}F_h{}^s=R_{kjhs}F_i{}^s,$$

$$R_{kjih}-R_{hijk}=\frac{1}{n-2}\{F_{kh}A_{(ji)}-F_{jh}A_{(ki)}+F_{ji}A_{(kh)}-F_{ki}A_{(jh)}$$

$$-F_{ih}(A_{kj}-2E_{kj})+F_{kj}(A_{ih}-2E_{ih})\},$$

$$R_{kjih}+R_{jikh}+R_{ikjh}$$

$$=\frac{2}{n-2}\{F_{kj}(A_{ih}-E_{ih})+F_{ji}(A_{kh}-E_{kh})+F_{ik}(A_{jh}-E_{jh})\},$$

$$nA_{ji}-4E_{ji}-2(n-2)V_{ji}+2(E-A)F_{ji}=0$$

and

$$D_lR_{kjih}+D_kR_{jlih}+D_jR_{lkih}$$

$$=\frac{2}{n-2}(F_{lk}D_sR_{tjih}+F_{kj}D_sR_{tlih}+F_{jl}D_sR_{tkih})F^{st}$$

$$-\frac{2}{n-2}(F_{lk}D_jE_{ih}+F_{kj}D_lE_{ih}+F_{jl}D_kE_{ih}).$$

§5. Complex conformal connections

In an almost complex manifold M with a torsion tensor $S_{ji}{}^h=-u^jF_{ji}$, we consider a conformal change of Hermitian metric

(5.1) $$\bar{g}_{ji}=e^{2p}g_{ji},\quad \bar{F}_{ji}=e^{2p}F_{ji},\quad \bar{F}_i{}^h=\bar{F}_{js}\bar{g}^{sh}=F_i{}^h,$$

where p is a scalar function, and we look for an affine connection $\bar{\Gamma}_{ji}{}^h$ such that

(5.2) $$\bar{D}_k\bar{g}_{ji}=0,\quad \bar{D}_k\bar{F}_{ji}=0,$$

where $\bar{D}_k$ are the operator of covariant differentiation with respect to the connection $\bar{\Gamma}_{ji}{}^h$, and the torsion tensor $\bar{S}_{ji}{}^h$ is given by

(5.3) $$\bar{S}_{ji}{}^h=-v^h\bar{F}_{ji},$$

where v^h are components of a vector field. We call such a metric change a complex conformal change of the metric.

By the remark above, we have, for the components $\bar{\Gamma}_{ji}{}^h$ of this affine connection,

(5.4) $$\bar{\Gamma}_{ji}{}^h=\overline{\begin{Bmatrix}h\\ji\end{Bmatrix}}+\bar{F}_j{}^hv_i+\bar{F}_i{}^hv_j-v^h\bar{F}_{ji},$$

where $\overline{\begin{Bmatrix}h\\ji\end{Bmatrix}}$ are the Christoffel symbols with $\bar{g}_{ji}$, and $v_i=v^s\bar{g}_{si}$.

Also, using

$$\overline{\begin{Bmatrix} h \\ ji \end{Bmatrix}} = \begin{Bmatrix} h \\ ji \end{Bmatrix} + p_j\delta_i^h + p_i\delta_j^h - p^h g_{ji}, \tag{5.5}$$

we have

$$\bar{\Gamma}_{ji}{}^h = \begin{Bmatrix} h \\ ij \end{Bmatrix} + p\delta_i^h + p_i\delta^h - p^h g_{ji} + v_j\bar{F}_i{}^h + v_i\bar{F}_j{}^h + v^h\bar{F}_{ji}, \tag{5.6}$$

where $p_i = \partial_i p$ *and* $p^h = p_s g^{sh}$. From (2.7), using (5.1), we have

$$\bar{\Gamma}_{ji}{}^h = \Gamma_{ji}{}^h + p_j\delta_i^h + p_i\delta_j^h - p^h g_{ji} + (v_j - u_j) F_i{}^h + (v_i - u_i) F_j{}^h - (e^{2p}v^h - u^h) F_{ji}. \tag{5.7}$$

If we define q^h by

$$q^h = e^{2p}v^h - u^h, \tag{5.8}$$

then

$$q_i = q^s g_{si} = (e^{2p}v^s - u^s) g_{si} = v^{2p}\bar{g}_{si} - u^s g_{si} = v_i - u_i. \tag{5.9}$$

Substituting (5.8) and (5.9) into (5.7), we have

$$\bar{\Gamma}_{ji}{}^h = \Gamma_{ji}{}^h + p_j\delta_i^h + p_i\delta_j^h - p^h g_{ji} + q_j F_i{}^h + q_i F_j{}^h - q^h F_{ji}. \tag{5.10}$$

We now compute $\bar{D}_k\bar{F}_{ji}$ and find

$$\begin{aligned} \bar{D}_k\bar{F}_{ji} &= \bar{D}_k(e^{2p}F_{ji}) \\ &= e^{2p}\{(q_sF_j{}^s - p_j) F_{ki} + (p_i - q_sF_i{}^s) F_{kj} \\ &\qquad - (q_i + p_iF_i{}^s) g_{kj} + (q_j + p_sF_j{}^s) g_{kj}\}. \end{aligned}$$

Thus, in order that $\bar{D}_k\bar{F}_{ji} = 0$, we must have

$$(q_sF_j{}^s - p_j) F_{ki} + (p_i - q_sF_i{}^s) F_{kj} - (q_i + p_sF_i{}^s) g_{kj} + (q_j + p_sF_j{}^s) g_{ki} = 0$$

for which, transvecting with g^{kj}, we find

$$(n-2)(p_sF_i{}^s + q_i) = 0,$$

that is, assuming $n \geq 4$

$$p_tF_i{}^t + q_i = 0.$$

Therefore,

$$q_i = -p_sF_i{}^s, \quad p_i = q_sF_i{}^s. \tag{5.11}$$

The converse being evident, we have

PROPOSITION 5.1. *In an almost complex manifold with a torsion tensor,* *by a complex conformal change of the metric*

$$\bar{g}_{ji}=e^{2p}g_{ji},\quad \bar{F}_{ji}=e^{2p}F_{ji},$$

the affine connection $\bar{\Gamma}_{ji}{}^{h}$ which satisfies

$$\bar{D}_k\bar{g}_{ji}=0,\quad \bar{D}_k\bar{F}_{ji}=0$$

and

$$\bar{\Gamma}_{ji}{}^{h}-\bar{\Gamma}_{ij}{}^{h}=-2v^h\bar{F}_{ji},$$

where p is a scalar function and v^h is a vector field, is given by

$$\bar{\Gamma}_{ji}{}^{h}=\Gamma_{ji}{}^{h}+p_i\delta_j^h+p_j\delta_i^h-p^hg_{ji}+q_jF_i{}^h+q_iF_j{}^h-q^hF_{ji}$$

where

$$p_i=\partial_i p,\quad p^h=p_sg^{sh},\quad q_i=-p_sF_i{}^s,\quad q^h=q_sg^{sh}.$$

We call such an affine connection a complex conformal connection in an almost complex manifold with a torsion tensor.

§6. Curvature tensor of a complex conformal connection and its invariant

We consider a complex conformal connection (5.10) in an almost complex manifold with a torsion tensor and compute the curvature tensor of $\bar{\Gamma}_{ji}{}^{h}$:

(6.1) $$\bar{R}_{kji}{}^{h}=\partial_k\bar{\Gamma}_{ji}{}^{hh}-\partial_j\bar{\Gamma}_{ki}{}^{h}+\bar{\Gamma}_{ks}{}^{h}\bar{\Gamma}_{ji}{}^{s}-\bar{\Gamma}_{js}{}^{h}\bar{\Gamma}_{ki}{}^{s}.$$

By a straightforward computation, we find

(6.2) $$\bar{R}_{kji}{}^{h}=R_{kji}{}^{h}+\delta_j^hp_{ki}-\delta_k^hp_{ji}-g_{ji}p_k{}^h+g_{ki}p_j{}^h$$
$$+F_j{}^hq_{ki}-F_k{}^hq_{ji}-F_{ji}q_k{}^h+F_{ki}q_j{}^h+F_i{}^h\alpha_{kj}-2F_{kj}\beta_i{}^h,$$

where

(6.3) $$p_{ji}=D_jp_i-p_jp_i+q_jq_i+(\lambda/2)g_{ji},\quad p_{ji}=p_{ij},$$

(6.4) $$\lambda=p_sp^s=q_sq^s,$$

(6.5) $$q_{ji}=-p_{js}F_i{}^s,\quad p_{ji}=q_{js}F_i{}^s,$$

(6.6) $$\alpha_{kj}=q_{kj}-q_{jk}-\mu F_{kj},\quad \alpha_{kj}=-\alpha_{jk}.$$

(6.7) $$\mu=\lambda+2q_su^s,$$

(6.8) $$\beta_{ji}=p_jq_i-p_iq_j+q_jw_i-q_iw_j+p_ju_i-u_jp_i,$$

(6.9) $$\beta_i^{\ h}=\beta_{is}g^{sh},\quad \beta_{ji}=-\beta_{ij}.$$

If we define

(6.10) $$\bar{R}_{kji}{}^s\bar{g}_{sh}=\bar{R}_{kjih}$$

(6.11) $$\bar{R}_{stji}\bar{F}^{st}=-2\bar{E}_{ji},\quad \bar{R}_{jist}\bar{F}^{st}=-2\bar{V}_{ji},\quad \bar{R}_{sjit}\bar{F}^{st}=\bar{A}_{ji},$$

then, from (4.8), we have

(6.12) $$\bar{E}_{ji}=E_{ji}-2(q_{ji}-q_{ij})-(\alpha/2)F_{ji}+n\beta_{ji},$$

where $\alpha=\alpha_{st}F^{st}=2p-n\mu$, $p=p_{st}g^{st}=q_{st}F^{st}$, and transvecting with F^{ji} $(=e^{2p}\bar{F}^{ji})$, we have

$$e^{2p}\bar{E}=E-4p+\frac{n}{2}(2\beta-\alpha),$$

where $\alpha=\alpha_{st}F^{st}=2p-n\mu$, $\beta=\beta_{st}F^{st}=2\mu$ and

(6.13) $$e^{2p}E=E-(n+4)p+\frac{n(n+4)}{2}\mu.$$

From (4.8), (6.11) and (6.2), we have

(6.14) $$\bar{V}_{ji}=V_{ji}-2(q_{ji}-q_{ij})+2\mu F_{ji}-(n/2)\alpha_{ji},$$

(6.15) $$\bar{A}_{ji}=A_{ji}-(n-1)q_{ji}+q_{ij}-pF_{ji}-\alpha_{ji}+2\beta_{ji}$$

and transvecting with $F^{ji}(=e^{2p}\bar{F}^{ji})$, we have

(6.16) $$e^{2p}\bar{V}=V-(n+4)p+\frac{n(n+4)}{2}\mu,$$

(6.17) $$e^{2p}\bar{A}=A-2(n+1)p+(n+4)\mu.$$

From (6.16) and (6.17), we find

(6.18) $$p=\frac{1}{2(n^2-4)}(nA-2E)-\frac{1}{2(n^2-4)}(n\bar{A}-2\bar{E})e^{2p},$$

(6.19) $$\mu=\frac{1}{(n+4)(n^2-4)}\{(n+4)A-2(n+1)E\}$$
$$-\frac{1}{(n+4)(n^2-4)}\{(n+4)\bar{A}-2(n+1)\bar{E}\}e^{2p}.$$

If we put

$$Q=\frac{1}{2(n^2-4)}(nA-2E),\qquad \bar{Q}=\frac{1}{2(n^2-4)}(n\bar{A}-2\bar{E}),$$

$$N=\frac{(n+4)A-2(n+1)E}{(n+4)(n^2-4)},\quad \bar{N}=\frac{(n+4)\bar{A}-2(n+1)\bar{E}}{(n+4)(n^2-4)},$$

(6.18) and (6.19) can be written as

$$(6.20)\qquad p=Q-\bar{Q}e^{2p},\quad \mu=N-\bar{N}e^{2p}$$

and consequently

$$(6.21)\qquad PF_{ji}=QF_{ji}-\bar{Q}\bar{F}_{ji},\quad \mu F_{ji}=NF_{ji}-\bar{N}\bar{F}_{ji}.$$

From (6.15), we have

$$(6.22)\qquad \bar{A}_{ji}=\bar{A}_{ij}=A_{ji}-A_{ij}-2(n+2)(q_{ji}-q_{ij})-(2P-2\mu)F_{ji}+4\beta_{ji},$$

from which, using (6.12) and (6.21), we have

$$(6.23)\qquad q_{ji}-q_{ij}=\frac{1}{(n+4)(n-2)}\{n(A_{ji}-A_{ij})-4E_{ji}-2(n-2)\bar{Q}F_{ji}\}$$
$$-\frac{1}{(n+4)(n-2)}\{n(\bar{A}_{ji}-A_{ji})-4\bar{E}_{ji}-2(n-2)\bar{Q}\bar{F}_{ji}\},$$

$$(6.24)\qquad \beta_{ji}=\frac{1}{2(n+4)(n-2)}\Big[4(A_{ji}-A_{ij})-2(n+2)_{ji}$$
$$+(n-2)\{2\bar{Q}-(n+4)N\}F_{ji}-\Big]\frac{1}{2(n+4)(n-2)}\Big[4(\bar{A}_{ji}-\bar{A}_{ij})$$
$$-2(n+2)\bar{E}_{ji}+(n-2)\{2\bar{Q}-(n+4)\bar{N}\}\bar{F}_{ji}\Big]$$

On the other hand, from (6.15), we have

$$(6.25)\qquad \bar{A}_{ji}+\bar{A}_{ij}=A_{ji}+A_{ij}-(n-2)(q_{ji}+q_{ij}),$$

from which, using (6.23), we find

$$(6.26)\qquad q_{ji}=\frac{1}{2(n+4)(n-2)}\{(n+4)(A_{ji}+A_{ij})+n(A_{ji}-A_{ij})$$
$$-4E_{ji}-2(n-2)QF_{ji}\}$$
$$-\frac{1}{2(n+2)(n-2)}\{(n+4)(\bar{A}_{ji}+\bar{A}_{ij})+n(\bar{A}_{ji}-\bar{A}_{ij})-4\bar{E}_{ji}-2(n-2)\bar{Q}\bar{F}_{ji}\}.$$

Substituting (6.23) into (6.6), we obtain

$$(6.27)\qquad \alpha_{ji}=\frac{1}{(n+4)(n-2)}\Big[n(A_{ji}-A_{ij})-4E_{ji}-(n-2)\{2Q+(n+4)N\}F_{ji}\Big]$$
$$-\frac{1}{(n+4)(n-2)}\Big[n(\bar{A}_{ji}-\bar{A}_{ij})-4\bar{E}_{ji}-(n-2)\{2\bar{Q}+(n+4)\bar{N}\}\bar{F}_{ji}\Big]$$

If we define M_{ji}, L_{ji}, B_{ji} and T_{ji} by

(6.28) $$M_{ji}=\frac{1}{2(n+4)(n-2)}\{(n+4)(A_{ji}+A_{ij})+n(A_{ji}-A_{ij})-4E_{ji}-2(n-2)QF_{ji}\},$$

(6.29) $$L_{ji}=M_{js}F_i^{\ s},$$

(6.30) $$B_{ji}=\frac{1}{2(n+4)(n-2)}\Big[4(A_{ji}-A_{ij})-2(n+2)E_{ji}+(n-2)\{2Q-(n+4)N\}F_{ji}\Big],$$

(6.30) $$T_{ji}=\frac{1}{(n+4)(n-2)}\Big[n(A_{ji}-A_{ij})-4E_{ji}-(n-2)\{2Q+(n+4)N\}F_{ji}\Big]$$

then, (6.24), (6.26) and (6.27) can be written as

(6.32) $$\beta_{ji}=B_{ji}-\overline{B}_{ji},\quad q_{ji}=M_{ji}-\overline{M}_{ji},\quad \alpha_{ji}=T_{ji}-\overline{T}_{ji}$$

and consequently

(6.33) $$p_{ji}=q_{js}F_i^{\ s}=(M_{js}-\overline{M}_{js})F_i^{\ s}=L_{ji}-\overline{L}_{ji}$$

(6.34) $$L_j^{\ h}=L_{js}F^{sh},\quad M_j^{\ h}=M_{js}F^{sh},\quad B_j^{\ h}=B_{js}F^{sh}$$

Substituting (6.32), (6.33) and (6.34) into (6.2), we have

(6.35) $$\begin{aligned}&\overline{R}_{kji}{}^h+\delta_j^h\overline{L}_{ki}-\delta_k^h\overline{L}_{ji}-\overline{g}_{ji}\overline{L}_k{}^h+\overline{g}_{ki}\overline{L}_j{}^h\\&+\overline{F}_j{}^h\overline{M}_{ki}-\overline{F}_k{}^h\overline{M}_{ji}-\overline{F}_{ji}\overline{M}_k{}^h+\overline{F}_{ki}\overline{M}_j{}^h\\&+\overline{F}_i{}^h\overline{T}_{kj}-2\overline{F}_{kj}\overline{B}_i{}^h\\=&R_{kji}{}^h+\delta_j^hL_{ki}-\delta_k^hL_{ji}-g_{ji}L_k{}^h+g_{ki}L_j{}^h\\&+F_j{}^hM_{ki}-F_k{}^hM_{ji}-F_{ji}M_k{}^h+F_{ki}M_j{}^h\\&+F_i{}^hT_{kj}-2F_{kj}B_i{}^h\end{aligned}$$

If we define $C_{kji}{}^h$ as

(6.36) $$\begin{aligned}C_{kji}{}^h=&R_{kji}{}^h+\delta_j^hL_{ki}-\delta_k^hL_{ji}-g_{ji}L_k{}^h+g_{ki}L_j{}^h\\&+F_j{}^hM_{ki}-F_k{}^hM_{ji}-F_{ji}M_k{}^h+F_{ki}M_j{}^h\\&+F_i{}^hT_{kj}-2F_{kj}B_i{}^h,\end{aligned}$$

then (6.35) reduces into

(6.37) $$\overline{C}_{kji}{}^h=C_{kji}{}^h.$$

We call such a tensor $C_{kji}{}^h$ a complex conformal curvature tensor in an almost complex manifold with a torsion tensor.

Thus, we have the following theorem.

THEOREM *In an n-dimensional* $(n \geq 4)$ *almost complex manifold with a torsion tensor, the complex conformal curvature tensor* $C_{kji}{}^h$ *defined by*

$$C_{kji}{}^h = R_{kji}{}^h + \delta_j^h L_{ki} - \delta_k^h L_{ji} - g_{ji} L_k{}^h + g_{ki} L_j{}^h + F_j{}^h M_{ki} - F_k{}^h M_{ji} - F_{ji} M_k{}^h + F_{ki} M_j{}^h + F_i{}^h T_{kj} - 2F_{kj} B_i{}^h$$

is an invariant under a complex conformal change of metric

$$\bar{g}_{ji} = e^{2p} g_{ji}, \quad \bar{F}_{ji} = e^{2p} F_{ji}.$$

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