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ON f-NONEXPANSIVE MAPS

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1. Introduction

Let (X, d) be a metric space and f a continuous selfmap of X. A selfmap g of X is said to be *f-nonexpansive* (*ε-f-nonexpansive* for some $\varepsilon>0$) if

(1) $$d(gx, gy) \leq d(fx, fy)$$

for all $x, y \in X$ (with $d(fx, fy)<\varepsilon$). If $d(gx, gy)<d(fx, fy)$ for any $x, y \in X$, $gx \neq gy$ (for any $x, y \in X$ with $gx \neq gy$, $d(fx, fy)<\varepsilon$ for some $\varepsilon>0$), then g is said to be *f-contractive* (*ε-f-contractive*) [9]. A selfmap g of X is called an *f-contraction* if there exists an $\alpha \in [0, 1)$ such that $d(gx, gy) \leq \alpha d(fx, fy)$ for any $x, y \in X$. When $f=1_X$, the identity map of X, those are reduced to usual nonexpansive maps, contractive maps, or contractions (cf. [5], [6], etc.)

Let g be a selfmap of X such that, for some $x \in X$, the sequence $\{g^n x\}$ of iterates has a subsequence which converges to a point $y \in X$. Then y is fixed if g is contrative, periodic if g is ε-contractive [5]. There are also corresponding generalizations for nonexpansive and ε-nonexpansive maps [6].

Our first purpose in this paper is to extend those important results of M. Edelstein to (ε-)f-nonexpansive or (ε-)f-contractive maps in Sections 2 and 3. In the proofs of main theorems, we make use of Edelstein's methods. Consequently, theorems on fixed points and periodic points are obtained, and some of main results in [9] is also extended.

In recent works of Dotson [3], [4], of Guseman and Peters [7], and of Talman [13], results concerning the existence of fixed points of nonexpansive maps on certain classes of compact nonconvex sets of metric (linear) spaces are obtained.

Our second purpose of this paper is to extend those results to f-nonexpansive maps. Our main tools are fixed point criteria for compact Hausdorff spaces in [13] and for metric spaces in [7]. In Section 4, f-nonexpansive maps on compact metric spaces are considered. Section 5 deals on weakly compact subsets of Banach spaces, and Section 6 on starshaped compact

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subsets of metric linear spaces.

2. *f*-nonexpansive maps

Let f be a continuous selfmap of X. Given a point $x_0\in X$ and a map g: $X\to X$ an *f-iteration* of x_0 *under* g is sequence $\{fx_n\}_{n=0}^{\infty}$ given recursively by the rule $fx_n=gx_{n-1}$ for $n\geq 1$. If $gX\subset fX$ then every point of X has an f-iteration under g (not necessarily unique).

Given a selfmap g of X, a point $y\in X$ is said to belong to the *g-closure* of X, $y\in X_f{}^g$, if there is a point $\eta_0\in X$ and an f-iteration $\{f\eta_n\}_{n=0}^{\infty}$ of η_0 such that a subsequence of $\{f\eta_n\}$ converges to y (cf. [6]).

A sequence $\{x_n\}_{n=0}^{\infty}$ in X is said to be *isometric* (ε-*isometric*) if $d(x_m, x_n)=d(x_{m+k}, x_{n+k})$ for all $m, n, k=0, 1, 2, \cdots$ (with $d(x_m, x_n)<\varepsilon$) [5]. A point $x_0\in X$ is said to *generate* an *f-isometric* (ε-*f-isometric*) *sequence under* g if there is an isometric (ε-isometric) f-iteration $\{fx_n\}$ of x_0 (cf. [6]).

Using those concepts we extend Edelstein's results in [6] as follows:

PROPOSITION 2.1. *If* $g:X\to X$ *is* ε-*f-nonexpansive and* $x_0\in X$ *with* $fx_0\in X_f{}^g$, *then there exists an f-iteration* $\{fx_n\}$ *of* x_0 *such that a subsequence* $\{fx_{mj}\}_{j=1}^{\infty}$ *converges to* fx_0

Proof. Since $fx_0\in X_f{}^g$, there exists a point $\eta_0\in X$ and an f-iteration $\{f\eta_n\}$ of η_0 such that $fx_0=\lim_{i\to\infty} f\eta_{n_i}$ for some $\{n_i\}$. If $f\eta_m=fx_0$ for some m, we have $d(g\eta_m, gx_0)\leq d(f\eta_m, fx_0)=0$ and, hence, $g\eta_m=gx_0$. Hence, $\{f\eta_n\}_{n=m}^{\infty}$ is an f-iteration of x_0, by putting $\eta_{m+k}=x_k$, $k\geq 1$, and $\{m_j\}=\{n_i-m\}$, $n_i>m$, is a sequence satisfying $\lim_{j\to\infty} fx_{mj}=fx_0$. Otherwise, let δ be a fixed number such that $0<\delta<\varepsilon$. Then an $i=i(\delta)$ exists so that

$$(2)\qquad d(fx_0, f\eta_{n_{i+j}})<\delta/4 \quad (j=0, 1, 2, \cdots).$$

For such an i and for arbitrary $k(k=1, 2, \cdots)$ we have

$$d(f\eta_{n_i}, f\eta_{n_{i+k}})\leq d(fx_0, f\eta_{n_i})+d(fx_0, f\eta_{n_{i+k}})<\delta/2.$$

Now for any f-iteration $\{fx_n\}$ of x_0, we obtain from (2) for $j=0$ and (1)

$$\delta/4>d(fx_0, f\eta_{n_i})\geq d(gx_0, g\eta_{n_i})=d(fx_1, f\eta_{n_i+1})$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\geq d(fx_{n_{i+k}-n_i}, f\eta_{n_{i+k}}).$$

Hence

$$d(fx_0, fx_{n_{i+1}-n_i})\leq d(fx_0, f\eta_{n_i})+d(f\eta_{n_i}, f\eta_{n_{i+1}})+d(f\eta_{n_{i+1}}, fx_{n_{i+1}-n_i})$$
$$<\delta/4+\delta/2+\delta/4=\delta.$$

We put $m_1=n_{i+1}-n_i$. Suppose that $m_1<m_2<\cdots<m_{j-1}$ are already defined

and that

$$d(fx_0, f\eta_{m_i}) \leq 1/2 \text{ min } \{d(fx_0, f\eta_m) \mid m=1, 2, \cdots, m_{i-1}\}, \quad i=2, 3, \cdots, j-1.$$

Then we choose $m_j = n_{l+1} - n_l$ where l is chosen as to satisfy (2) with δ replaced by $1/2 \text{ min } \{d(fx_0, f\eta_m) \mid m=1, 2, \cdots, m_{j-1}\}$. Now the sequence $\{m_j\}$ so defined satisfies the requirements of the proposition.

THEOREM 2.2. *If $g: X \to X$ is ε-f-nonexpansive then each $x_0 \in X$ with $fx_0 \in X_f^g$ generates an ε-f-isometric sequence under g.*

Proof. Choose an f-iteration $\{fx_n\}$ of x_0 satisfying Proposition 2.1. Suppose there exist indices m, n and k such that $d(fx_m, fx_n) < \varepsilon$ and

$$\delta = d(fx_m, fx_n) - d(fx_{m+k}, fx_{n+k}) \neq 0.$$

Then we have

$$(3) \qquad d(fx_m, fx_n) - d(fx_{m+l}, fx_{n+l}) \geq \delta > 0 \quad (l \geq k)$$

for

$$\varepsilon > d(fx_m, fx_n) \geq d(gx_m, gx_n) = d(fx_{m+1}, fx_{n+1})$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\geq d(fx_{m+l}, fx_{n+l}).$$

Also from (1) and Proposition 2.1 it follows that for some $\{n_j\}$ and all l

$$\lim_{j\to\infty} f(x_l)_{n_j} = \lim_{j\to\infty} fx_{n_j+l} = fx_l.$$

Hence positive integer i exists such that $j \geq i$ implies

$$d(fx_{m+n_j}, fx_m) < \delta/2 \text{ and } d(fx_{n+n_j}, fx_n) < \delta/2.$$

However,

$$d(fx_m, fx_n) \leq d(fx_m, fx_{m+n_j}) + d(fx_{m+n_j}, fx_{n+n_j}) + d(fx_{n+n_j}, fx_n)$$
$$< \delta/2 + d(fx_{m+n_j}, fx_{n+n_j}) + \delta/2,$$

which contadicts to (3) for $n_j \geq \max(n_i, k)$. This shows that $\delta = 0$ and our proof is complete.

As an immediate consequence of Theorem 2.2, we obtain the corresponding statement concerning f-nonexpansive maps.

THEOREM 2.3. *If $g: X \to X$ is f-nonexpansive then each $x \in X$ with $fx \in X_f^g$ generates an f-isometric sequence under g.*

In case $f = 1_X$ Theorems 2.2 and 2.3 are reduced to results of Edelstein [6].

3. f-contractive maps

In this section, we consider some applications to f-contractive maps. We need the following.

LEMMA 3.1. ([9], *Lemma* 2.1) *Let f and g be commuting selfmaps of a metric space X. If g^N is f-contractive for some integer $N>0$ and f, g^N have a coincidence $\zeta\in X$, then $f\zeta$ is the unique common fixed point of f and g.*

LEMMA 3.2. *Let f be a continuous selfmap of a metric space X, $g:X\to X$, and g^N be f-contractive for some integer $N>0$. Then f and g^N have a coincidence $x_0\in X$ iff x_0 generates an f-isometric sequence under g^N.*

Proof. Suppose $fx_0=g^Nx_0$ for some $x_0\in X$. Then there is an f-iteration $\{fx_n\}_{n=0}^{\infty}$ of x_0 under g^N such that $x_n=x_0$ for all n. It is readily seen that $\{fx_n\}$ is an f-isometric sequence.

Conversely, if $x_0\in X$ generates an f-isometric sequence $\{fx_n\}_{n=0}^{\infty}$, then $d(fx_0,\ fx_1)=d(fx_1,\ fx_2)$. Suppose $g^Nx_0=fx_1\neq fx_2=g^Nx_1$. Then

$$d(fx_1,\ fx_2)=d(g^Nx_0,\ g^Nx_1)<d(fx_0,\ fx_1),$$

which is a contradiction. Hence we have $fx_1=fx_2$, which implies $fx_0=g^Nx_0$.

THEOREM 3.3. *A continuous selfmap f of a metric space X has a fixed point iff there is a map $g:X\to X$ commuting with f such that g^N is f-contractive for some $N>0$ and there is a point $x_0\in X$ generating an f-isometric sequence under g^N. Indeed, f and g have a unique common fixed point fx_0.*

Proof. Suppose that $f\eta=\eta$ for some $\eta\in X$. Define $g:X\to X$ by $gx=\eta$ for all $x\in X$. Then the necessity follows trivially. The converse follows from Lemmas 3.1 and 3.2.

From Theorems 2.3 and 3.3, we have

COROLLARY 3.4. *A continuous selfmap f of X has a fixed point iff there is a map $g:X\to X$ commuting with f such that g^N is f-contractive for some $N>0$ and $X_f{}^{g^N}\neq\phi$. Indeed, $\eta\in X_f{}^{g^N}$ is a unique common fixed point of f and g.*

In case $f=1_X$, Corollary 3.4 implies that a point of $X^g=X_{1_X}{}^g$ is fixed under g if g is contractive [5]. Furthermore, Theorem 3.5 and Corollaries 3.6, 3.7, 3.8 of [9] follow from Corollary 3.4. Note that Corollary 3.8 of [9] generalizes results of Rakotch [10], of Boyd-Wong [2], and of Jungck [8].

Suppose $g^NX\subset fX$ in Corollary 3.4. Since $X_f{}^{g^N}\neq\phi$ for a compact space X, we have

COROLLARY 3.5 *A continuous selfmap f of a compact metric space X has a fixed point iff there is a map $g: X \to X$ commuting with f such that g^N is f-contractive for some $N>0$ and $g^N X \subset fX$. Indeed, f and g have a unique common fixed point.*

Theorem 3.4 of [9] is a consequence of Corollary 3.5.

Note that Corollaries 3.4 and 3.5 also follow from Theorem 2.2 and Corollary 2.3 of [9].

For an ε-f-contractive map we have the following.

THEOREM 3.6. *Let f be a continuous selfmap of X. If $g: X \to X$ is ε-f-contractive, then for any $x_0 \in X$ with $fx_0 \in X_f{}^g$ there exists an f-iteration $\{fx_n\}_{n=0}^{\infty}$ of x_0 and an integer $j>0$ such that $fx_0 = fx_j$.*

Proof. Since g is ε-f-nonexpansive, by Proposition 2.1, x_0 has an f-iteration $\{fx_n\}$ such that there exist $i, j>0$ satisfying $d(fx_0, fx_i)<\varepsilon/2$, $d(fx_0, fx_{i+j})<\varepsilon/2$. Note that $\{fx_n\}$ is ε-f-isometric from the proof of Theorem 2.2. Since

$$d(fx_i, fx_{i+j}) \leq d(fx_0, fx_{i+j}) < \varepsilon,$$

we have

$$d(fx_i, fx_{i+j}) = d(fx_{i+1}, fx_{i+j+1}).$$

Suppose $fx_i \neq fx_{i+j}$. Since g is ε-f-contractive, we have

$$d(fx_{i+1}, fx_{i+j+1}) = d(gx_i, gx_{i+j}) < d(fx_i, fx_{i+j}),$$

which is a contradiction. Therefore we have $d(fx_i, fx_{i+j}) = 0 < \varepsilon$ and, hence, $d(fx_0, fx_j) = d(fx_i, fx_{i+j}) = 0$.

Theorem 3.6 generalizes the fact that if g is ε-contractive and $x \in X^g$ then x is a periodic point of g [5].

4. f-nonexpansive maps of compact metric spaces

We adopt the following modification of the main theorem of Talman [13].

THEOREM 4.1. *A continuous selfmap g of a compact Hausdoff space X has a fixed point iff there is a family $\mathfrak{F}$ of selfmaps of X satisfying*

(i) 1_X *is in the uniform closure of $\mathfrak{F}$, and*

(ii) *gh or hg has a fixed point in x for each h in $\mathfrak{F}$.*

Proof. If g has a fixed point, then $\mathfrak{F}=\{1_X\}$ satisfies (i) and (ii). For the converse, simply follow the proof of Theorem 1 of [13].

COROLLARY 4.2. *A continuous selfmap g of a compact Hausdorff space X has a fixed point iff there is an equicontinuous family $\mathfrak{F}$ of selfmaps of X satisfying*

(i) *1_X is in the pointwise closure of $\mathfrak{F}$, and*

(ii) *gh or hg has a fixed point in X for each h in $\mathfrak{F}$.*

Note that Guseman and Peters [7] essentially obtained Theorem 4.1 and Corollary 4.2 for compact metric spaces and also gave examples which show that certain hypothesis of 4.1 and 4.2 can not be relaxed.

They also obtained the following result [7] as a generalization of Smart's result [12].

COROLLARY 4.3. *If the identity map of a compact metric space X is the pointwise limit of contractive selfmaps of X, then each nonexpansive selfmap of X has a fixed point point.*

Now we have the following main result in this section.

THEOREM 4.4. *Let X be a compact metric space and f a continuous selfmap of X. If 1_X is the pointwise limt of contractive selfmaps of X commuting with f, then each f-nonexpansive selfmap g of X commuting with f has a fixed point.*

Proof. Let $\{h_n\}$ be a sequence of contractive selfmaps X which commute with f. By a result of Edeltein [5] or Corollary 3.4, each h_n has a unique fixed point x_n in X. Moreover, $hx_n=x_n$ implies $fh_nx=fx_n=h_nfx_n$, and hence we have $fx_n=x_n$. Therefore the set F of fixed points of f in X is nonempty, and by the continuity of f, also compact. If g is f-nonexpansive and commutes with f, it is immediate that g maps F into itself and g is nonexpansive on F. New by applying Corollary 4.3 to F we obtain our result.

In case $f=1_X$ theorem 4.4 is reduced to Corollary 4.3, and hence they are equivalent.

COROLLARY 4.5. *Let X be a compact metric space and f a continuous selfmap of X. Suppose there is a map $F:X\times[0,1]\to X$ which satisfies*

(1) $\lim_{t\to 1} F(x,t)=x$ *for any* $x\in X$,

(2) $d(F(x,t),\ F(y,t))<d(x,y)$ *for any* $x,y\in X$, $x\neq y$, *and any* $t\in[0,1)$, *and*

(3) $F(fx,\ t)=fF(x,t)$ *for each* $x\in X$, $t\in[0,1)$.

Then each f-nonexpansive selfmap of X commuting with f has a fixed point.

Proof. By (1), 1_X is in the pointwise closure of $\mathfrak{F}=\{h_t\}$ where $t\in[0,1)$ and $h_t=F(\cdot,t)$. By (2) h_t is contractive for any $t\in[0,1)$, and by (3) h_t commutes with f. Therefore, by Theorem 4.4 our proof is complete.

Note that in case $f=1_X$, Corollary 4.5 is reduced to a result of Guseman-Peters [7] and extend results in Talman [13] and Dotson [3], [4].

A metric space X is called an *S-space* if there exists an $x_0\in X$ such that for every $t\in(0,1)$ there is a contractive selfmap h_t of X satisfying

$$d(h_t x,\ x)\le(1-t)d(x_0,x)$$

for every $x\in X$ [1].

THEOREM 4.6. *Let f be a continuous selfmap of a compact S-space X such that $h_t f=fh_t$. Then any f-nonexpansive selfmp of X which commutes with f has a fixed point.*

Proof. Note that $\mathfrak{F}=\{h_t\,|\,t\in(0,1)\}$ is a net which converges to 1_X when $(0,1)$ is equipped with its usual order. Now our result follows from Theorem 4.4.

In case $f=1_X$ Theorem4.6 is reduced to a result of Baron and Matkowski [1].

5. f-nonexpansive maps on weakly compact sets

In this section we consider f-nonexpansive maps on weakly compact subsets of a Banach space. We need the following result of Jungck [8] or Corollary 3.4.

LEMMA 5.1. *A continuous selfmap f of a complete metric space X has a fixed point iff there is an f-contraction $g:X\to X$ commuting with f such that $gX\subset fX$. Indeed, f and g have a unigue common fixed point.*

THEOREM 5.2. *Let X be a weakly compact subset of a Banach space, and f a continuous selfmap of X. Suppose there is a map $F:X\times[0,1]\to X$ which satisfies*

(1) $\lim_{t\to 1}F(x,t)=x$ *for each* $x\in X$,

(2) *there is a selfmap ϕ of $(0,1)$ such that for every $x,y\in X$ and for every $t\in(0,1)$, we have*

$$\|F(x,t)-F(y,t)\|\le\phi(t)\|x-y\|,\ \text{and}$$

(3) $F(fx,\ t)=fF(x,t)$ *for every* $x\in X,\ t\in(0,1)$.

Then each f-nonexpansive map $g:X\to X$ commuting with f such that $gX\subset fX$ has a fixed point.

Proof. For each $n=1,2,\cdots$, let $t_n=n/(n+1)$ and define $h_n=F(\cdot,t)$. Then h_n converges uniformly to 1_X. In view of Theorem 4.1, it suffices to show that gh_n has a fixed point for any n. For any $x,y\in X$, we have

$$\|g(h_nx)-g(h_ny)\|\leq\|f(h_nx)-f(h_ny)\|=\|h_n(fx)-h_n(fy)\|\leq\phi(t_n)\|fx-fy\|.$$

Since $\phi(t_n)<1$ for every n, gh_n is an f-contraction on X relative to the norm. But X is weakly compact, hence norm closed, and hence norm complete. Since gh_n commutes with f and $gh_nX\subset fX$, gh_n has a fixed point by Lemma 5.1.

In case $f=1^X$, 5.2 is reduced to results of Dotson [3], [4] and of Talman [13].

6. f-nonexpansive maps on starshaped sets

Let E be a metric linear space with translation invariant metric d [11]. A subset X of E is said to be *starshaped* if there exists $x_0\in X$ such that $tx+(1-t)x_0\in X$ for every $t\in[0,1]$, $x\in X$. Let θ denote the zero element of E. A metric d for E is said to be strictly monotone if $d(\theta, tx)<d(\theta,x)$ for every $x\neq\theta$ and $t\in[0,1)$.

THEOREM 6.1. *Let E be a metric linear space with strictly monotone metric d. Let X be a compact subset of E starshaped at $x_0\in X$ and f a continuous selfmap of X satisfying $f(tx+(1-t)x_0)=tfx+(1-t)x_0$ for any $x\in X$ and $t\in[0,1)$. Then every f-nonexpansive selfmap of X which commutes with f has a fixed point.*

Proof. Define a map $F:X\times[0,1]\to X$ by $F(x,t)=tx+(1-t)x_0$. Then for each $x\in X$, $\lim_{t\to 1}F(x,t)=x$ and $F(fx,t)=tfx+(1-t)x_0=fF(x,t)$ for each $x\in X$, $t\in[0,1)$. Given $x,y\in X$, $x\neq y$, and $t\in[0,1)$, we have

$$\begin{aligned} d(F(x,t),\ F(y,t))&=d(tx,\ ty)=d(\theta,\ t(x-y))\\ &<d(\theta,x-y)=d(x,y). \end{aligned}$$

Therefore, from Corollary 4.5 our proof is complete.

A *p-norm* $(0\leq p\leq 1)$ on a linear space E is a nonnegative function $\|\ \|$ on $X\times X$ which satisfies $\|x\|=0$ iff $x=\theta$, $\|x+y\|\leq\|x\|+\|y\|$, $\|\lambda x\|=|\lambda|^p\|x\|$ for each $x,y\in X$ and each scalar λ. Since each p-norm generates a translation invariant metric $d(x,y)=\|x-y\|$ which is strictly monotone, we have the following from 6.1.

COROLLARY 6.2. *Let E be a p-normed space, X a compact subset of E starshaped at $x_0\in X$, and f a continuous selfmap of X satisfying*

$$f(tx+(1-t)x_0)=tfx+(1-t)x_0$$

for any $x\in X$ *and* $t\in[0,1)$. *Then every f-nonexpansive selfmap of X which commutes with f has a fixed point.*

In case where d is not strictly monotone, we have the following.

THEOREM 6.3. *Let E be a metric linear space, and suppose that* $d(\theta, tx)\leq d(\theta,x)$ *whenever* $|t|\leq 1$. *Let X be a compact subset of E starshaped at* $x_0\in X$ *and f a continuous selfmap of X satisfying* $f(tx+(1-t)x_0)=tfx+(1-t)x_0$ *for any* $x\in X$ *and* $t\in[0,1)$. *If a selfmap g of X commutes with f on X and satisfies* $d(tgx, tgy)\leq d(tfx, tfy)$ *for any* $x,y\in X$ *and* $t\in[0,1]$, *then g has a fixed point.*

Proof. The function $\rho(x,y)=\int_0^1 d(tx, ty)dt$ *is* an equivalent metric on X. Since g is nonexpansive with respect to ρ, the result follows from 6.1.

In case $f=1_X$, 6.1, 6.2 and 6.3 are reduced to results of Guseman-Peters [7].

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