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A DUALITY OF GLOBAL *M*-SPECTRAL MAXIMAL SPACES FOR A TWO-DECOMPOSABLE OPERATOR

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1. Introduction

Auther introduced the notion of the *local* and the *global* M-spectral maximal spaces for a decomposable operator (see [7]). In this note we consider its duality. Throughout this note let X be the complex Banach space, $B(X)$ the algebra of all bounded linear operators acting on X, and we denote X^* the dual of X, $T^*\in B(X^*)$ the dual operator of $T\in B(X)$.

1.1. An invariant subspace Y for T is called a spectral maximal space, for T if Y contains all invariant subspaces Z for which $\sigma(T|Z)\subseteq\sigma(T|Y)$.

1.2. $T\in B(X)$ is decomposable if for any finite open covering $\{G_1, G_2, \cdots G_n\}$ of $\sigma(T)$ there exists a system $\{Y_1, Y_2, \cdots, Y_n\}$ of spectral maximal spaces for T such that

(i) $\sigma(T|Y_i)\subset G_i$ $(i=1,2,\cdots,n)$, and

(ii) $X=Y_1+Y_2+\cdots+Y_n$

It is called two-decomposable if $n=2$. The following facts are known:

(a) If T is two-decomposable, then T^* is also two-decomposable.

(b) If X is reflexive, then T is two-decomposable if and only if T^* is two-decomposable (see [13]).

Let σ be a separate part of $\sigma(T)$ and let T be decomposable. Then $T|E(\sigma, T)X$ is also a decomposable operator, where

$$E(\sigma, T)=\frac{1}{2\pi i}\int_\Gamma^{-1}(\lambda I-T)\,d\lambda,$$

Γ is any contour surrounding σ and separates σ and $\sigma'=\sigma(T)\setminus\sigma$ (see [6], p. 35). It is also well known that

(1) $E(\sigma,T)X$ is a spectral maximal space of T and $\sigma(T|E(\sigma,T)X)=\sigma$.

(2) If T is two-decomposable, then $X_T(F)=\{x\in X|\sigma_T(x)\subseteq F\}$ is a spectral maximal space for T and $\sigma(T|X_T(F))\subseteq F\cap\sigma(T)$ for every closed set F in $\mathbf{C}$.

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2. Global M-spectral maximal spaces for a two-decomposable operator

We define a global M-spectral maximal space as follows:

2.1 DEFINITION. A closed subspace $M \subsetneqq X$ is said to be a global M-spectral maximal space for T if

(G_1) M is a spectral maximal space for T, and

(G_2) If there exists a spectral maximal space Y for T such that $M \subsetneqq Y \subseteq X$, than $Y=X$.

For more details see [7].

The following proposition was proved (see [7]) for a decomposable operator, in fact, it can be shown directly for two-decomposable operator.

2.2 LEMMA. *Let T be a two-decomposable operator with the spectrum does not reduced to one point. A global M-spectral maximal space for T exists if and only if there exists an isolated point in $\sigma(T)$.*

Proof. Suppose that M is a global M-spectral maximal space of T. Then, from the definition, $\sigma(T|M) \subsetneqq \sigma(T)$; For, if $\sigma(T|M)=\sigma(T)$ then $M=X$ since M is a spectral maximal space for T and X is the trivial spectral maximal space for T. Thus $\sigma(T) \setminus \sigma(T|M) \neq \phi$. We put $\sigma(T) \setminus \sigma(T|M)=\sigma$, σ is a separate part (or open and closed subset) in $\sigma(T)$, $\sigma(T)=\sigma(T|M) \cup \sigma$ and $\sigma(T|M) \cap \sigma=\phi$.

Now, we will claim that σ is an isolated point in $\sigma(T)$, that is σ is a singleton. Supposs σ is not a single point, we choose a point $z_0 \in \sigma$ so that $z_0 \notin \partial(\sigma(T))$ (bounary of $\sigma(T)$) and

$$\sigma(T|M) \subsetneqq \sigma(T) \cup \{z_0\} \subsetneqq \sigma(T). \tag{2.1}$$

Setting $F=\sigma(T|M) \cup \{z_0\}$, F is closed in $\mathbf{C}$ and $\sigma(T|M) \cap \{z_0\}=\phi$. Since T is two-decomposable, both sets $\sigma(T|M)$ and $\{z_0\}$ are closed subsets in $\mathbf{C}$, it follows that

$$X_T(F)=X_T(\sigma(X|M)) \oplus X_T(\{z_0\}) \tag{2.2}$$

Moreover, $X_T(\{z_0\})$ can not be zero vector. For, if $\{0\}=X_T\{z_0\}$ then $\phi=\sigma_T(0)=\{z_0\}$, a contradition. The first equality comes from the facts that every two-decomposable operator has the single valued extension property, and that $\sigma_T(x)=\phi$ if and only if $x=0$. Therefore, we have

$$M=X_T(\sigma(T|M)) \subsetneqq X_T(F). \tag{2.3}$$

It follows from (2.1), (2.3) and $\sigma(T)=\bigcup_{x\in X}\sigma_T(x)$ that

$$M\subsetneqq X_T(F)\subsetneqq X_T(\sigma(T))=X. \tag{2.4}$$

Therefore, there exists a spectral maximal subspace $X_T(F)$ for T which majorizes M. This contradicts the hypothesis for M.

For the sufficiency, suppose there exists an isolated point z_0 in $\sigma(T)$. If we put $\sigma=\sigma(T)\setminus\{z_0\}$, σ must be a closed subset in $\mathbf{C}$. We may choose closed jordan curves Γ_1 and Γ_2 lying in the resolvent set $\rho(T)$ and surrounding σ and $\{z_0\}$ respectively such that no point in common. Hence we can difine projection operators $E(\sigma, \mathrm{T})$ and $E(\{z_0\}, T)X$ with

$$E(\sigma, T)X\oplus E(\{z_0\}, T)X=X. \tag{2.5}$$

Putting $E(\sigma, T)X=M$, M is the global M-spectral maximal space for T: For, if there exists a spectral maximal subspace Y of T such that $M\subsetneqq Y\subseteq X$, then we have $\sigma(T|M)=\sigma\subsetneqq\sigma(T|Y)\subseteq\sigma(T)$. If $\sigma(T|Y)\neq\sigma(T)$ then $\sigma(T|Y)\setminus\sigma\subsetneqq\sigma(T)\setminus\sigma=\{z_0\}$, Whence $\sigma(T|Y)/\sigma=\phi$. But this is impossible since $\sigma\subsetneqq\sigma(T|Y)$. This completes the proof.

2.3 PROPOSITION. *Let T be two-decomposable, let M be a global M-spectral maximal space for T and z_M be the corresponding isolated point in $\sigma(T)$. Then $\sigma(T_M)=\{z_M\}$. Where T_M is the quotient operator induced by T on X/M and is defined by $T_M(\pi x)=\pi(Tx)$ endowed with the norm*

$$\|T_M\|=\sup_{\|\pi x\|\leq 1}\|T_M(\pi x)\|,\quad \|\pi x\|=\inf_{z\in\pi x}\|z\|.$$

For the proof see [7].

3. A dual of the global M-spectral maximal space

The following proposition is due to S. Frunza (see [13]).

3.1 PROPOSITION. *Let T be a two-decomposable operator and F be a closed subset of $\mathbf{C}$. Then $X_T^*(F)$ is a spectral maximal space for T^*. Moreover*

$$\sigma(T^*|X_T^*(F))\subseteq F\cap\sigma(T^*),$$

where $X_T^*(F)=X_T(\mathbf{C}\setminus F)^{\perp}=\{x^*\in X^*: x^*(X_T(\mathbf{C}\setminus F))=0\}$.

3.2 THEOREM. *Let T be a two-decomposable operator, let M be a global M-spectral maximal space for T. Then the global M-spectral maximal space for T^* exists and it has the form $X_{T^*}^*(\sigma(T|M)=X_T(\{z_M\})^{\perp}$.*

Proof. Since T is two-decomposable and M is a spectral maximal space for T, we have $M=X_T(\sigma(T|M))$.

Let z_M be the corresponding isolated point in $\sigma(T)$, then

$$M = X_T(\sigma(T|M)) = X_T(\sigma(T)\setminus\{z_M\}).$$

From the proposition 3.1, we see that

(3.1) $M^{\perp} = X^*_{T^*}(\{z_M\})$ and

$$X^*_{T^*}(\sigma(T|M)) = X_T(\sigma(T)\setminus\sigma(T|M))^{\perp} = X_T(\{z_M\})^{\perp}.$$

Since $\sigma_T(x)=\phi$ if and only if $x=0$ and $(X/M)^* = M^{\perp}$ for every closed subspace M of X, we have $X_T(\phi)=\{0\}$ whence $X_T(\phi)^{\perp}=X^*$. It follows that

$$X_T(\{z_M\})^{\perp} \subsetneqq X_T(\phi)^{\perp} = X^*_{T^*}(\mathbf{C}) = X^*, \tag{3.2}$$

that is,

$$X^*_T\ (\sigma(T|M)) \subsetneqq X^*.$$

Suppose now that Y^* is a spectral maximal space for T^* such that $X^*_T(\sigma(T|M)) \subsetneqq Y^* \subseteq X^*$. Since T^* is also a two-decomposable operator, the spectral maximal space Y^* can be represented by $X^*_{T^*}(F) = Y^*$ for some closed subset of $\mathbf{C}$. Both $Y^*_{T^*}(\sigma(T|M))$ and $Y^* = X^*_{T^*}(F)$ are spectral maximal space of T^*, it follows that

$$\sigma(T|M) \subsetneqq F \cap \sigma(T^*) \subseteq \sigma(T^*) = \sigma(\mathrm{T}). \tag{3.3}$$

Thus $\sigma(T)\cap F = \sigma(T)$ since $\sigma(T)\setminus\sigma(T|M) = \{z_M\}$ (single point). So we have

$$X^*_{T^*}(\sigma(T)\cap F) = X^*_{T^*}(\sigma(T)) = X^*, \text{ or } X^*_{T^*}(F) = X^*. \tag{3.4}$$

Hence $X^*_{T^*}(\sigma(T|M))$ is the global M-spectral maximal space for T^*.

From the above facts we have the following immediate consequences:

3.3 COROLLARY. *Let T be two-decomposable. If a global M-spectral maximal space for T exists, then a global M-spectral maximal space for T^* exists.*

3.4 COROLLARY. *Let T be two-decomposable and let X be reflexive. A global M-spectral maximal space for T exists if and only if a global M-spectral maximal space for T^* exists.*

The Riesz decomposition theorem says that if $\sigma(T) = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are disjoint non-empty closed sets, then T has the complementary pair $\{M_1, M_2\}$ of non-trivial invariant subspaces such that $\sigma(T|M_1) = \sigma_1$ and $\sigma(T|M_2) = \sigma_2$.

For the dual operator of a two-decomposable operator, we have the following dual results.

3.5 THEOREM. *Let T be two-decomposable. If σ_1 is the non-empty separate part of $\sigma(T)$, then T^* has a pair of spectral maximal spaces $X^*_{T^*}(\sigma_1)$ and $X^*_{T^*}(\sigma_2)$ such that*

$$\sigma(T^* \mid X^*_{T^*}(\sigma_i)) = \sigma_i \quad (i=1,2) \text{ and } X^*_{T^*}(\sigma_1) \oplus X^*_{T^*}(\sigma_2) = X^*,$$

where $\phi \neq \sigma_2 = \sigma(T) \setminus \sigma_1$

Proof. We notice that $X_T(\mathbf{C}) = X_T(\sigma(T)) = X$ since $\sigma(T) = \bigcup_{x \in X} \sigma_T(x)$. Therefore, the equalities $X^*_{T^*}(F) = X_T(\mathbf{C} \setminus F)^\perp = X_T(\sigma(T) \setminus F)$ hold for every closed subset F in $\mathbf{C}$. Hence, without loss of generality, we may assume $F \subset \sigma(T)$. If we put $\sigma_2 = \sigma(T) \setminus \sigma_1$, then both σ_1 and σ_2 are open and closed subsets of $\sigma(T)$, which are separate parts of $\sigma(T)$. Thus $Y_i = X_T(\sigma_i)$, $i=1,2$ is spectral maximal space for T and $X_T(\sigma_i) = E(\sigma_i)X$ $(i=1,2)$. Therefore, we have

$$\sigma(T \mid X_T(\sigma_i)) = \sigma_i \quad (i=1,2).$$

The equalities

$$X^*_{T^*}(\sigma(T) \setminus \sigma_1) = X_T(\sigma(T) \setminus \sigma_2)^\perp \quad \text{and} \quad X^*_{T^*}(\sigma(T) \setminus \sigma_2) = X(\sigma(T) \setminus \sigma_1)^\perp$$

imply that

$$X^*_{T^*}(\sigma_2) = X_T(\sigma_1)^\perp, \quad X^*_{T^*}(\sigma_1) = X_T(\sigma_2)^\perp.$$

Since T^* is also a two-decomposable operator, each $X^*_{T^*}(\sigma_i)$ $(i=1,2)$ is spectral maximal space for T^* with $X^*_{T^*}(\sigma_1) \oplus X^*_{T^*}(\sigma_2) = X^*$.

Now, we claim that the equality $\sigma(T^* \mid X^*_{T^*}(\sigma_i)) = \sigma_i$ holds. For this purpose, we put $Y_i = X_T(\sigma_i)$ $(i=1,2)$. Since $T^* Y_i^\perp \subset Y_i^\perp$ for $i=1,2$, we can define the restriction operator $T^* \mid Y_i^\perp$ $(i=1,2)$. The identifications

$$(X/Y_1)^* = Y_1^\perp, \quad (X/Y_2)^* = Y_2^\perp$$

will imply that the operator $(T_{Y_i})^* : (X/Y_i)^* \to (X/Y_i)^*$
may be indentified with the operator $T^* \mid Y_i^\perp : Y_i^\perp \to Y_i^\perp$. Moreover, from the fact that $\sigma(T_Y) = \overline{\sigma(T) \setminus \sigma(T \mid Y)}$ for any spectral maximal space Y for T and that $\sigma(T) = \sigma(T^*)$, it follows that

$$\begin{aligned} \sigma(T^* \mid X^*_{T^*}(\sigma_1)) &= \sigma(T^* \mid X_T(\sigma_2)^\perp) = \sigma((T_{Y_2})^*) \\ &= \sigma(T_{Y_2}) = \overline{\sigma(T) \setminus \sigma(T \mid Y_2)} = \sigma_1. \end{aligned}$$

Similary, $\sigma(T^* \mid X^*_{T^*}(\sigma_2)) = \sigma_2$. This completes the proof.

From Theorem 3.2 and Theorem 3.5, we have the follwing

3.6 COROLLARY. *Let T be two-decomposable with the spectrum does not reduced to one point. If $\sigma(T)$ contains an isolated point, then T^* has the global*

M-spectral maximal space $X^*_{T^*}(\sigma(T|M))$ and a spectral maximal space $X^*_{T^*}(\{z_M\})$ such that

$$\sigma(T^*|X^*_{T^*}(\sigma(T|M)))=\sigma(T|M),$$

$$\sigma(T^*|X^*_{T^*}(\{z\}))=\sigma(T|X_T(\{z_M\}))=\{z_M\},$$

and

$$X^*_{T^*}(\sigma(T|M))\oplus X^*_{T3}(\{z_M\})=X^*,$$

where M is the global M-spectral maximal space for T and z_M is the corresponding isolated point.

4. Examples

(1) Suppose X is a complex Banach space with dim $X=\infty$ and $T\in B(X)$ is a compact oparator, so this is decomposable (see [6], p. 33). Since $\sigma(T)$ is at most countable and has at most one limit point, namely 0. Moreover, each non-zero $\lambda\in\sigma(T)$ is an eigen value of T and also an isolated point in $\sigma(T)$. Therefore each $M_\lambda=X_T(\sigma(T)\setminus\{\lambda\})\ (\lambda\neq0)$ is a global M-spectral maximal space for T with $\sigma(T|M_\lambda)=\sigma(T)\setminus\{\lambda\}$.

And each $X^*_{T^*}(\sigma(T|M_\lambda))\ (\lambda\neq0)$ is also a global M-spectral maximal space for T^*. Moreover since T is compact if and only if T^* is compact and a global M-spectral maximal space for T exists if and only if a global M-spectral maximal space exists for T^*. There are at most countable number of global M-spectral maximal spaces for T and T^*. Furthermore, $M_\lambda{}^\perp=X^*_{T^*}(\{\lambda\})$ and $\sigma(T^*|M_\lambda{}^\perp)=\{\lambda\}$ for $0\neq\lambda\in\sigma(T)$.

(2) We consider the right shift operator S and the multiplication operator M on l^2 space defined by

$$(Sx)(n)=\begin{cases}0 & \text{if } n=0\\ x(n-1) & \text{if } n\geq1\end{cases}$$

and

$$M(x)(n)=(n+1)^{-1}x(n) \text{ if } n\geq0$$

respectively for any $x=\{\xi_0,\xi_1,\xi_2,\cdots\}\in l^2$ with $\sum_{i=0}^{\infty}|\xi_i|^2<\infty$.

Define the operator $T=MS$ by

$$(Tx)(n)=\begin{cases}0 & \text{if } n=0\\ (n+1)^{-1}x(n-1) & \text{if } n\geq1\end{cases}\qquad \text{for each } x\in l^2.$$

It is easily be shown that T is a compact operator and T has no eigen-value. Since $\lim_{n\to\infty}\|T^n\|^{\frac{1}{n}}=0$, $\sigma(T)$ consists of exactly one point 0, thus the operator T has no global M-spectral maximal space.

(3) Let X be the set of all bounded complex functions defined on **C** with the norm $\|x\|=\sup|x(z)|$. Define addition and multiplication in the usual way. This makes X into a commutative Banach algebra with the unit. Let T be the multiplication operator on X defined by

$$(Tx)(z)=a(z)x(z), \text{ where } a(z)=\begin{cases} z & \text{if } |z|\leqslant 1 \\ 1 & \text{if } |z|>1 \end{cases}$$

Then the spectrum of T is the closed unit disc centered at the origin in the complex plane. Hence no isolated point exists in $\sigma(T)$. In addition since every multiplication operator on a Banach algebra is decomposable (see [6], p. 201], there is no global M-spectral maximal space for T.

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