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ON SEMIDIRECT FACTORS OF A LOCALLY COMPACT GROUP

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1. It is well known that a connected locally compact abelian group G is isomorphic with $\mathbf{R}^n \times E$, where $\mathbf{R}^n$ is a n-dimensional vector group and E a connected compact group. This means, in a sense, that the complimentary part of the maximal compact subgroup is determined uniquely.

Now suppose that a locally compact group G is a semidirect product of a connected subgroup N and by a compact subgroup C. Is the complimentary part N of C unique in the sense that, if $G=N'C$ is another way of decomposition as above, N and N' are isomorphic (as topological groups)? We shall show that this is the case if N is, in addition, nilpotent. However, we will give an example that N need not be unique even if it is solvable.

2. As usual, $\mathbf{C}$ and $\mathbf{R}$ will denote the set of complex numbers and real numbers, respectively. Let $T=\{t\in\mathbf{C} : |t|=1\}$ be a circle and $H=\mathbf{C}\times\mathbf{C}\times\mathbf{R}$. Now, let η: $T\times T\rightarrow \mathrm{Aut}(H)$ be a continuous homomorphism defined by $\eta(t_1t_2)(z_1, z_2, r)=(t_1z_1, t_1z_2, r)$ and let G be the semidirect product of H by $T\times T$ determined by η. For each homomorphism $f:\mathbf{R}\rightarrow T\times T$, a subgroup N_f of G will be defined by $N_f=\{z_1, z_2, r, f(r) \mid z_i\in\mathbf{C},\ i=1,2,\ r\in\mathbf{R}\}$. Since N_f is a extension of commutative group by a commutative group, it is solvable. Therefore, subgroups $N_1=\{z_1, z_2, r, e^{2\pi ir}, 1\}$ and $N_2=\{z_1, z_2, r, e^{2\pi ir}, e^{-2\pi ir}\}$, which corresponds to homomorphisms $f_1 : r\longrightarrow(e^{2\pi ir}, 1)$ and $f_2 : r\longrightarrow(e^{2\pi ir}, e^{-2\pi ir})$, respectively, are solvable. Direct computation shows that the typical elements of $[N_1, N_1]$ and $[N_2, N_2]$ are, respectively, of the form

$$(z_1(1-e^{2\pi ir'})+z_1'(e^{2\pi ir}-1), 0, 0, 1, 1) \quad \text{and}$$
$$(z_1(1-e^{-2\pi ir'})+z_1'(e^{2\pi ir}-1),\ z_2(1-e^{2\pi ir'})+z_2'(e^{-2\pi ir}-1), 0, 1, 1).$$

Therefore, $[N_1, N_1]=\mathbf{C}$ and $[N_2, N_2]=\mathbf{C}\times\mathbf{C}$. This means N_1 and N_2 are not isomorphic, whereas $G=N_i\cdot H$, $N_i\cap H=\{e\}$ and $H=\{(0, 0, 0, t_1, t_2) : (t_1, t_2)\in T\times T\}$.

3. We begin with the following

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LEMMA. *Let N be a connected nilpotent locally compact group. Then any compact subgroup C of N is central in N.*

Proof. Let U be any neighborhood of identity element e of N and K a compact normal subgroup of N such that N/K is a analytic group. For the moment assume that any compact subgroup of a nilpotent analytic group is central. Now, CK/K is central and hence we have $[C, N]\subseteq[CK, N]\subseteq K\subseteq U$. Since U is arbitrary, $[C, N]=\{e\}$. Therefore the problem is reduced to the case N is a analytic group. It is well known that the quotient group of an analytic group N modulo its center Z is simply connected. Therefore CZ/Z, a compact subgroup of a simply connected nilpotent analytic group, is trivial. Hence C is a central subgroup.

THEOREM. *Let G be a locally compact group and $G=N_i\cdot C$ $(i=1,2)$ be a semidirect product of a connected nilpotent normal subgroup N_i by a compact subgroup C. Then there exist a isomorphism $h: G\to G$ which fixes every point of C. In particular, N_1 and N_2 are isomorphic.*

Proof. Let N be the nilradical of G. There are uniquely determined continuous projections f_i and g_i such that $x=f_i(x)g_i(x)$ $(i=1,2)$. Define a continuous map $h: G\to G$ by setting $h(x)=f_2(f_1(x))\cdot g_1(x)$. We shall show that the map h is a homomorphism of G. The following equalities are an easy consequence of the definitions:

$$\begin{aligned} h(xy) &= f_2(f_1(xy))g_1(xy) \\ &= f_2(f_1(x)g_1(x)f_1(y)g_1(x)^{-1})g_1(xy) \\ &= f_2(f_1(x))g_2(f_1(x))f_2(g_1(x)f_1(y)g_1(x)^{-1})g_2(f_1(x))^{-1}g_1(xy). \end{aligned}$$

Since $f_1(x)=f_2(f_1(x))g_2(f_1(x))$, $f_2(f_1(x))^{-1}f_1(x)=g_2(f_1(x))\in N\cap K$ is a central element of N. Since $f_2(g_1(x)f_1(y)g_1(x)^{-1})\in N$, we have

$$h(xy)=f_2(f_1(x))f_2(g_1(x)f_1(y)g_1(x)^{-1}g_1(xy). \tag{1}$$

On the other hand,

$$\begin{aligned} g_1(x)f_1(y)g_1(x)^{-1} &= g_1(x)(f_2(f_1(y))g_2(f_1(y))g_1(x)^{-1} \\ &= g_1(x)f_2(f_1(y))g_1(x)^{-1}(g_1(x)g_2(f_1(y))g_1(x)^{-1}) \end{aligned}$$

and hence $f_2(g_1(x)f_1(y)g_1(x)^{-1}=g_1(x)f_2(f_1(y))g_1(x)^{-1}$. Consequently, by (1) we have $h(xy)=h(x)h(y)$. Similarly, the map h' defined by $h'(x)=f_1(f_2(x))g_2(x)$ is a continuous homomorphism which is the inverse of h. To show this, let $x, y\in G$. To make the computation simpler, we note that $f_1(f_2(x))=f_1(x)$ and $f_2(f_1(x))=f_2(x)$ which is a consequence of the uniqueness of representation of an element as a product.

Now we have

$$\begin{aligned} h'(h(x)) &= h'(f_2(x)g_1(x)) = f_1(f_2(x)g_1(x))g_2(f_2(x)g_1(x)) \\ &= f_1(f_2(x)g_1(x))g_2(g_1(x)) \\ &= f_1(f_2(x)g_1(x))g_1(x) \\ &= f_1(f_1(f_2(x))g_1(f_2(x)g_1(x))g_1(x) \\ &= f_1(x)g_1(x) = x. \end{aligned}$$

This implies that h' is the inverse of h and hence that h is a homeomorphism.

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