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PROPERTIES OF ALMOST C-CONTINUOUS FUNCTIONS

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1. Introduction

In 1970, Gentry and Hoyle [1] have introduced the concept of c-continuous functions which has been investigated by Long and Hendrix [4] and Long and Herrington [5]. In 1975, Long and Hamlett [3] have defined and studied the concept of H-continuity analogous to that of c-continuity. On the other hand, in 1968 Singal and Singal [8] have introduced a weak form of continuity called almost-continuity. Quite recently, Suk Geun Hwang [9] has introduced a new class of functions, called almost c-continuous functions, which contains the class of c-continuous functions and that of almost-continuous functions. The purpose of the present paper is to continue the investigation of almost c-continuous functions.

2. Preliminaries

Throughout the present paper spaces mean always topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *regular closed* (*regular open*) if $\mathrm{Cl}(\mathrm{Int}(A))=A$ (resp. $\mathrm{Int}(\mathrm{Cl}(A))=A$).

DEFINITION 2.1. A function $f:X\to Y$ is said to be *c-continuous* [1] if for each $x\in X$ and each open neighborhood V of $f(x)$ in Y such that $Y-V$ is compact, there exists an open neighborhood U of x in X such that $f(U)\subset V$.

A subset S of a space X is said to be *quasi H-closed relative to* X [7] (simply *quasi H-closed*) if for every cover $\{V_\alpha \mid \alpha\in\nabla\}$ of S by open sets of X, there exists a finite subfamily ∇_0 of ∇ such that

$$S\subset \cup\{\mathrm{Cl}(V_\alpha)\mid \alpha\in\nabla_0\}.$$

DEFINITION 2.2. A function $f:X\to Y$ is said to be *H-continuous* [3] if for each $x\in X$ and each open neighborhood V of $f(x)$ such that $Y-V$ is quasi H-closed, there exists an open neighborhood U of x such that $f(U)\subset V$.

DEFINITION 2.3. A function $f:X\to Y$ is said to be *almost-continuous* [8] if

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for each $x \in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$.

DEFINITION 2.4. A function $f: X \to Y$ is said to be *almost c-continuous* [9] if for each $x \in X$ and each open neighborhood V of $f(x)$ such that $Y - V$ is compact, there exists an open neighborhood U of x such that $f(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$.

Almost-continous functions are almost c-continuous but the converse is not true in general [9, Example]. The following theorem shows the relationships between the functions defined above.

THEOREM 2.5. *The following implications hold and none of these implications can, in general, be reversed:*

continuity $\Longrightarrow$ *H-continuity* $\Longrightarrow$ *c-continuity* $\Longrightarrow$ *almost c-continuity.*

Proof. See [3, Example 3; Example 4] and [9, Example].

3. Strongly-closed graphs.

Let $f: X \to Y$ be a function of a space X into a space Y. The subset $\{(x, f(x)) \mid x \in X\}$ of the product space $X \times Y$ is called the graph of f and usually denoted by $G(f)$.

DEFINITION 3.1 The graph $G(f)$ is said to be *strongly-closed* [6] if for each $(x, y) \notin G(f)$, there exist open sets $U \subset X$ and $V \subset Y$ containing x and y, respectively, such that $[U \times \mathrm{Cl}(V)] \cap G(f) = \phi$.

The following lemma is a useful characterization of functions with strongly-closed graphs.

LEMMA 3.2 (Long and Herrington [6]). *The graph $G(f)$ is strongly-closed if and only if for each $(x, y) \notin G(f)$, there exist open sets $U \subset X$ and $V \subset Y$ containing x and y, respectively, such that $f(U) \cap \mathrm{Cl}(V) = \phi$.*

THEOREM 3.3 *If a function $f: X \to Y$ has a strongly-closed graph, then it is H-continuous.*

Proof. Suppose that $G(f)$ is strongly-closed. Let K be any quasi H-closed set of Y and $x \notin f^{-1}(K)$. For each $y \in K$, $(x, y) \notin G(f)$ and hence, by Lemma 3.2, there exist open sets $U_y(x) \subset X$ and $V(y) \subset Y$ containing x and y, respectively, such that $f(U_y(x)) \cap \mathrm{Cl}(V(y)) = \phi$. Now, the family $\{V(y) \mid y \in K\}$ is a cover of K by open sets of Y. Hence there exists a finite subset K_0 of K such that $K \subset \cup \{\mathrm{Cl}(V(y)) \mid y \in K_0\}$. Put $U = \cap \{U_y(x) \mid y \in K_0\}$. Then U is an open set of X containing x and $U \cap f^{-1}(K) = \phi$. This shows that $f^{-1}(K)$ is a closed set of X. Therefore, it follows from Theorem 1 of [3] that f is H-continuous.

THEOREM 3.4. *If Y is a locally compact Hausdorff space and $f: X \to Y$ is an almost c-continuous function, then $G(f)$ is strongly-closed.*

Proof. Let $(x, y) \notin G(f)$. Then $y \neq f(x)$ and hence there exist disjoint open sets V_1 and V_2 containing y and $f(x)$, respectively. Since Y is locally compact Hausdorff, there exists an open set V of Y such that $y \in V \subset \mathrm{Cl}(V) \subset V_1$ and $\mathrm{Cl}(V)$ is compact. Since $\mathrm{Cl}(V)$ is regular closed and compact in Y, the almost c-continuity of f implies that $f^{-1}(\mathrm{Cl}(V))$ is closed in X [9, Theorem 1]. Put $U = X - f^{-1}(\mathrm{Cl}(V))$. Then U is an open set containing x and $f(U) \cap \mathrm{Cl}(V) = \phi$. Hence it follows from Lemma 3.2 that $G(f)$ is strongly-closed.

As an immediate consequence of Theorem 2.5, Theorem 3.3 and Theorem 3.4 we have

COROLLARY 3.5. *Let Y be a locally compact Hausdorff space. Then for a function $f: X \to Y$, the following are equivalent:*

(1) *$G(f)$ is strongly-closed.*
(2) *f is H-continuous.*
(3) *f is c-continuous.*
(4) *f is almost c-continuous.*

THEOREM 3.6. *If Y is a compact (compact Hausdorff) space and $f: X \to Y$ is an almost c-continuous function, then f is almost-continuous* (resp. *continuous*).

Proof. Let F be any regular closed set of Y. Since Y is compact, F is compact and hence $f^{-1}(F)$ is closed in X [9, Theorem 1]. Therefore, it follows from Theorem 2.2 of [8] that f is almost-continuous. If Y is compact Hausdorff, then it is regular and hence f is continuous.

COROLLARY 3.7. *Let Y be a compact Hausdorff space. Then, for a function $f: X \to Y$, the following are all equivalent:*

(1) *f is continuous.*
(2) *f is almost-continuous.*
(3) *$G(f)$ is strongly-closed.*
(4) *f is H-continuous.*
(5) *f is c-continuous.*
(6) *f is almost c-continuous.*

Proof. It is known that an almost-continuous function into a Hausdorff space has a strongly-closed graph [6, Theorem 1]. Hence this is an immediate consequence of Theorem 3.6.

Theorem 13 of [4] states that if $f: X \to Y$ is an almost-continuous bijection and Y is a Hausdorff space, then $f^{-1}: Y \to X$ is c-continuous. We shall show that "*almost-continuous*" in this result can be replaced by "*weakly-continuous*". A function $f: X \to Y$ is said to be *weakly-continuous* [2] if for each $x \in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(U) \subset \mathrm{Cl}(V)$. Every almost-continuous function is weakly-continuous but the converse is not true in general [8, Example 2.3].

LEMMA 3.8 (Levine [2]). *A function* $f: X \to Y$ *is weakly-continuous if and only if* $f^{-1}(V) \subset \mathrm{Int}[f^{-1}(\mathrm{Cl}(V))]$ *for every open set* V *of* Y.

LEMMA 3.9. *If* $f: X \to Y$ *is a weakly-continuous function and* K *is a compact set of* X, *then* $f(K)$ *is quasi H-closed relative to* Y.

Proof. Let $\{V_\alpha | \alpha \in \nabla\}$ be any cover of $f(K)$ by open sets of Y. By Lemma 3.8, we have $K \subset \cup \{\mathrm{Int}(f^{-1}(\mathrm{Cl}(V_\alpha))) | \alpha \in \nabla\}$. Since K is compact, there exists a finite subfamily ∇_0 of ∇ such that

$$K \subset \cup \{\mathrm{Int}(f^{-1}(\mathrm{Cl}(V_\alpha))) | \alpha \in \nabla_0\}.$$

Therefore, we obtain $f(K) \subset \cup \{\mathrm{Cl}(V_\alpha) | \alpha \in \nabla_0\}$. This shows that $f(K)$ is quasi H-closed in Y.

THEOREM 3.10. *If* Y *is a Hausdorff space and* $f: X \to Y$ *is a weakly-continuous bijection, then* $f^{-1}: Y \to X$ *is c-continuous.*

Proof. Let K be any compact set of X. Then by Lemma 3.9 $f(K)$ is quasi H-closed relative to Y. Since Y is Hausdorff, $(f^{-1})^{-1}(K) = f(K)$ is closed in Y [7, (2.5), p. 161]. Therefore, it follows from Theorem 1 of [1] that f is c-continuous.

4. Product spaces.

Let $\{Y_\alpha | \alpha \in \nabla\}$ be any family of spaces and $\prod Y_\alpha$ denote the product space. It is known that if Y_α is locally compact Hausdorff and $f_\alpha: X \to Y_\alpha$ is c-continuous for each $\alpha \in \nabla$, then a function $f: X \to \prod Y_\alpha$ defined by $f(x) = \{f_\alpha(x)\}$ is c-continuous [5, Theorem 2.1]. The following theorem is an improvement of this result.

THEOREM 4.1. *If* Y_α *is a locally compact Hausdorff space and* $f_\alpha: X \to Y_\alpha$ *is an almost c-continuous function for each* $\alpha \in \nabla$, *then a function* $f: X \to \prod Y_\alpha$, *defined by* $f(x) = \{f_\alpha(x)\}$ *for each* $x \in X$, *is H-continuous.*

Proof. Let $(x, y) \notin G(f)$. Then $y \neq f(x)$ and there exists $\beta \in \nabla$ such that $y_\beta \neq f_\beta(x)$. Since Y_β is locally compact Hausdorff and $f_\beta: X \to Y_\beta$ is almost c-continuous, $G(f_\beta)$ is strongly-closed by Theorem 3.4. Hence, by Lemma 3.2 there exist open sets $U \subset X$ and $V_\beta \subset Y_\beta$ containing x and y_β, respectively, such that $f_\beta(U)\cap \mathrm{Cl}(V_\beta)=\phi$. Put $V=V_\beta\times \prod_{\alpha\neq\beta} Y_\alpha$, then V is an open set containing y and $f(U)\cap \mathrm{Cl}(V)=\phi$. Hence, by Lemma 3.2, $G(f)$ is strongly-closed. It follows from Theorem 3.3 that f is H-continuous.

COROLLARY 4.2. *If X is Hausdorff, Y is locally compact Hausdorff and $f:X\rightarrow Y$ is almost c-continuous, then the graph function $g:X\rightarrow X\times Y$, defined by $g(x)=(x, f(x))$ for each $x\in X$, is H-continuous.*

Proof. The identity function $i_X:X\rightarrow X$ is continuous and X is Hausdorff. Hence it follows from Corollary of [6] that $G(i_X)$ is strongly-closed. The proof is quite similar to that of Theorem 4.1.

THEOREM 4.3. *If Y_α is a locally compact Hausdorff space and $f_\alpha:X_\alpha\rightarrow Y_\alpha$ is an almost c-continuous function for each $\alpha\in\nabla$, then a function $f:\prod X_\alpha\rightarrow\prod Y_\alpha$, defined by $f(\{x_\alpha\})=\{f_\alpha(x_\alpha)\}$ for each $\{x_\alpha\}\in\prod X_\alpha$, is H-continuous.*

Proof. Let $(x, y)\notin G(f)$. Then $y\neq f(x)$ and there exists $\beta\in\nabla$ such that $y_\beta\neq f_\beta(x_\beta)$. Since Y_β is locally compact Hausdorff and f_β is almost c-continuous, $G(f_\beta)$ is strongly-closed by Theorem 3.4. Hence, by Lemma 3.2, there exist open sets $U_\beta\subset X_\beta$ and $V_\beta\subset Y_\beta$ containing x_β and y_β, respectively, such that $f_\beta(U_\beta)\cap \mathrm{Cl}(V_\beta)=\phi$. Put

$$U=U_\beta\times\prod_{\alpha\neq\beta}X_\alpha \text{ and } V=V_\beta\times\prod_{\alpha\neq\beta}Y_\alpha.$$

Then U and V are open sets containing x and y, respectively, such that $f(U)\cap \mathrm{Cl}(V)=\phi$. This shows that $G(f)$ is strongly-closed. Therefore, it follows from Theorem 3.3 that f is H-continuous.

5. Compact spaces.

For any space (Y, σ) the family $\mathbf{B}$ of regular open sets having compact complements forms a base for a new topology σ^* on Y. The reason is that if U and V belong to $\mathbf{B}$, then $U\cap V$ is regular open and $Y-(U\cap V)=(Y-U)\cup(Y-V)$ is compact. Let $f:X\rightarrow(Y,\sigma)$ be a function and $f^*:X\rightarrow(Y, \sigma^*)$ a function defined by $f^*(x)=f(x)$ for each $x\in X$. Then, it is obvious that f is almost c-continuous if and only if f^* is continuous. Since $\sigma^*\subset\sigma$, the identity function $i:(Y,\sigma)\rightarrow(Y,\sigma^*)$ is continuous and also $i^{-1}:(Y,\sigma^*)\rightarrow(Y,\sigma)$ is almost c-continuous.

THEOREM 5.1. *For any space (Y,σ), the space (Y,σ^*) is compact.*

Proof. Let $\{V_\alpha\mid\alpha\in\nabla\}$ be any σ^*-open cover of Y. Let $y\in Y$. Then there exist an $\alpha_0\in\nabla$ and $V\in\mathbf{B}$ such that $y\in V\subset V_{\alpha_0}$. Since $Y-V$ is compact in (Y,σ), there exists a finite subfamily ∇_0 of ∇ such that $Y-V\subset\cup\{V_\alpha\mid\alpha\in$

$\nabla_0\}$. Therefore, we have $Y = V_{\alpha_0} \cup [\cup \{V_\alpha | \alpha \in \nabla_0\}]$. This shows that (Y, σ^*) is compact.

THEOREM 5.2. *If* (Y, σ) *is a compact Hausdorff space, then* (Y, σ^*) *is Hausdorff and* $\sigma^* = \sigma$.

Proof. Let y_1 and y_2 be a pair of distinct points of Y. Since (Y, σ) is Hausdorff, there exist disjoint σ-open sets V_1 and V_2 containing y_1 and y_2, respectively. Therefore, we have $\mathrm{Int}(\mathrm{Cl}(V_1)) \cap \mathrm{Int}(\mathrm{Cl}(V_2)) = \phi$ and $y_j \in V_j \subset \mathrm{Int}(\mathrm{Cl}(V_j))$, where $j=1,2$. Since (Y, σ) is compact, $Y - \mathrm{Int}(\mathrm{Cl}(V_j))$ is compact and $\mathrm{Int}(\mathrm{Cl}(V_j)) \in \sigma^*$. This shows that (Y, σ^*) is Hausdorff. Since compact Hausdorff spaces are minimal Hausdorff, we have $\sigma \subset \sigma^*$ and hence $\sigma = \sigma^*$.

THEOREM 5.3. *If* (Y, σ^*) *is Hausdorff, then* (Y, σ) *is compact and* $\sigma^* = \sigma$.

Proof. Let $\{V_\alpha | \alpha \in \nabla\}$ be any σ-open cover of Y. Since (Y, σ^*) is Hausdorff, there exist disjoint σ^*-open sets V_1 and V_2 such that $Y - V_j$ is compact in (Y, σ) for $j=1,2$. Hence there exists a finite subfamily ∇_j of ∇ such that

$$Y - V_j \subset \cup \{V_\alpha | \alpha \in \nabla_j\}, \quad \text{where } j=1,2.$$

Therefore, we obtain

$$Y = (Y - V_1) \cup (Y - V_2) = \cup \{V_\alpha | \alpha \in \nabla_1 \cup \nabla_2\}.$$

Hence (Y, σ) is compact. Since $\sigma^* \subset \sigma$ and (Y, σ^*) is Hausdorff, (Y, σ) is Hausdorff and hence minimal Hausdorff. Therefore, we obtain $\sigma^* = \sigma$.

COROLLARY 5.4. *A space* (Y, σ) *is compact Hausdorff if and only if the space* (Y, σ^*) *is compact Hausdorff.*

Proof. This is an immediate consequence of Theorem 5.2 and Theorem 5.3.

COROLLARY 5.5. *If a function* $f: X \to (Y, \sigma)$ *is almost c-continuous and* (Y, σ^*) *is Hausdorff, then* f *is continuous.*

Proof. Since (Y, σ^*) is Hausdorff, (Y, σ) is compact Hausdorff by Theorem 5.3. Hence it follows from Theorem 3.6 that f is continuous.

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