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# CODIMENSION 2 SUBMANIFOLDS WITH A 2-QUASI-UMBILICAL NORMAL DIRECTION

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## §1. Introduction.

Let $M^n$ be an $n$-dimensional submanifold of an $(n+2)$-dimensional Riemannian manifold $M^{n+2}(c)$ of constant curvature $c$. It seems interesting to find out some precise relations between the second fundamental tensors $(h_{ji})$ and $(k_{ji})$ of $M^n$ w.r.t. two orthonormal normal vector fields $\xi$ and $\xi^{\perp}$. In [3], Bang-yen Chen and Kentaro Yano proved that if $(h_{ji})$ has only one eigenvalue and $\xi$ is non-parallel (in the normal bundle), then $(k_{ji})$ has an eigenvalue of multiplicity $\geqq n-1$. In [6], Bang-yen Chen and one of the authors proved that if $(h_{ji})$ has an eigenvalue of multiplicity $n-1$, then in generic case $(k_{ji})$ has an eigenvalue of multiplicity $\geqq n-3$.

In the present paper, we'll compute the precise form of $(k_{ji})$ whenever $(h_{ji})$ has an eigenvalue of multiplicity $n-2$. In particular, we'll prove that in generic case $(k_{ji})$ has an eigenvalue of multiplicity $\geqq n-5$. As an application we'll give a result on spherical foliations.

## §2. Preliminaries

Let $M^n$ be an $n$-dimensional submanifold of an $(n+2)$-dimensional space form $M^{n+2}(c)$ of curvature $c$*). Let $\tilde{g}$ and $\tilde{\nabla}$ be respectively the metric tensor of $M^{n+2}(c)$ and the corresponding operator of covariant differentitation. Let $g$ and $\nabla$ be the induced metric, respectively connection, on $M^n$. The second fundamental form $h$ of $M^n$ is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{1}$$

where $X$ and $Y$ are tangent vector fields on $M^n$. $h(X, Y)$ is a normal vector field on $M^n$ which is symmetric on $X$ and $Y$. For a normal vector field $\eta$ on $M^n$ we write

$$\tilde{\nabla}_X \eta = -A_\eta(X) + D_X \eta \tag{2}$$

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*) Manifolds, functions, ⋯ are supposed to be sufficiently differentiable, and the dimensions $n$ are supposed to be sufficiently high to keep our discussions meaningful.

where $-A_\eta(X)$ and $D_X\eta$ denote respectively the tangential and normal components of $\tilde{\nabla}_X\eta$. $A_\eta$ is the *second fundamental tensor* of $M^n$ w.r.t. $\eta$ and $D$ is the normal connection of $M^n$ in $M^{n+2}(c)$. We have

$$\langle A_\eta(X), Y\rangle = \langle h(X, Y), \eta\rangle \tag{3}$$

where $\langle\ ,\ \rangle$ denotes the scalar product in $M^{n+2}(c)$.

Let $\{x^k\}$ be a local coordinate system in $M^n$ (indices $k, j, i, h, t, s$ run over the range $\{1, 2, \cdots, n\}$), and put $\partial_i = \frac{\partial}{\partial x^i}$. Let $\xi$ and $\xi^\perp$ be two orthogonal normal sections of $M^n$. Then

$$D_j = l_j\xi^\perp,\quad D_j\xi^\perp = -l_j\xi \tag{4}$$

where $D_j = D_{\partial_j}$ and $l_j$ is the *third fundamental tensor*. $\xi$ and $\xi^\perp$ are said to be parallel or non-parallel (in the normal bundle) according as the third fundamental tensor vanishes or never vanishes identically.

In the following we'll denote the second fundamental tensors w.r.t. $\xi$ and $\xi^\perp$ respectively by $h_{ji}$ and $k_{ji}$. Then the *equations of Gauss, Codazzi and Ricci* are respectively

$$K_{kjih} = c(g_{kh}g_{ji} - g_{jh}g_{ki}) + h_{kh}h_{ji} - h_{jh}h_{ki} + k_{kh}k_{ji} - k_{jh}k_{ki}; \tag{5}$$

$$\nabla_k h_{ji} + l_j k_{ki} = \nabla_j h_{ki} + l_k k_{ji}, \tag{6}$$

$$\nabla_k k_{ji} - l_j h_{ki} = \nabla_j k_{ki} - l_k h_{ji}; \tag{7}$$

$$\nabla_j l_i - \nabla_i l_j = h_{it}k_j^{\ t} - h_{jt}k_i^{\ t}, \tag{8}$$

where $K_{kjih}$ is the Riemann-Christoffel curvature tensor of $M^n$, and $h_j^{\ i} = h_{jt}g^{ti}$ and $k_j^{\ i} = k_{jt}g^{ti}$.

If on $M^n$ there exist two functions $\alpha, \beta$ and a unit vector field $u_j$ such that

$$h_{ji} = \alpha g_{ji} + \beta u_j u_i, \tag{9}$$

then $M^n$ is said to be *quasi-umbilical* w.r.t. the normal direction $\xi$. (9) is equivalent to the statement that w.r.t. $\xi$ $M^n$ has a *principal curvature with multiplicity* $\geq n-1$. If respectively identically $\alpha=0$, $\beta=0$ or $\alpha=\beta=0$ then $M^n$ is said to be *cylindrical*, *umbilical* or *geodesic* w.r.t. $\xi$. A quasi-umbilical normal direction $\xi$ which is not umbilical is said to be *properly quasi-umbilical*; in this case, the tangent direction of $M^n$ determined by $u_j$ is called the *distinguished direction* w.r.t $\xi$. $M^n$ is said to be a *totally quasi-umbilical* submanifold of $M^{n+2}(c)$ if $M^n$ is quasi-umbilical w.r.t. two orthogonal normal directions [1]. For a survey on quasi-umbilical submanifolds, see [5].

Now, we say that $M^n$ is *2-quasi-umbilical* w.r.t. $\xi$ if on $M^n$ there exist three functions $\alpha, \beta, \gamma$ and two orthogonal unit vector fields $u_j$ and $v_j$ such that

$$h_{ji}=\alpha g_{ji}+\beta u_j u_i+\gamma v_j v_i. \tag{10}$$

This is equivalent to say that w. r. t. $\xi$ $M^n$ has a *principal curvature with multiplicity* $\geqq n-2$. (Of course, for this definition to be meaningful it is essential that $n>4$.) In the following, when dealing with 2-quasi-umbilical normal directions $\xi$, it will be understood that both $\beta$ and $\gamma$ don't vanish, in other words that $\xi$ is *properly 2-quasi-umbilical*. The two tangent directions of $M^n$ determined by $u_j$ and $v_j$ are called the *distinguished directions* w. r. t. $\xi$. $M^n$ is said to be a *totally 2-quasi-umbilical* submanifold of $M^{n+2}(c)$ if $M^n$ is 2-quasi-umbilical w. r. t. 2 orthogonal normal directions. In the obvious way one may define $p$-quasi-umbilical normal directions and totally $p$-quasi-umbilical submanifolds, for $2<p<n-1$.

## § 3. Submanifolds with a non-parallel 2-quasi-umbilical normal direction

The main purpose of this paragraph is to give expressions for the second fundamental tensor $k_{ji}$ w. r. t. $\xi^{\perp}$ when the second fundamental tensor w. r. t. $\xi$ is given by

$$h_{ji}=\alpha g_{ji}+\beta u_j u_i+\gamma v_j v_i \tag{11}$$

and $\xi$ is non-parallel.

The procedures of calculations being of the same type as written in a detailed way in [6], here we'll only sketch the computations for the case where $l_j$ doesn't belong to the plane spanned by the distinguished directions $u_j$ and $v_j$ and is neither orthogonal to $u_j$ nor to $v_j$; for other cases we'll just state results.

First one inserts (11) into (6).

$$\begin{aligned}&\alpha_k g_{ji}+\beta_k u_j u_i+\gamma_k v_j v_i+\beta u_j \nabla_k u_i+\beta u_i \nabla_k u_j+\gamma v_j \nabla_k v_i+\gamma v_i \nabla_k v_j+l_j k_{ki}\\&=\alpha_j g_{ki}+\beta_j u_k u_i+\gamma_j v_k v_i+\beta u_k \nabla_j u_i+\beta u_i \nabla_j u_k+\gamma v_k \nabla_j v_i+\gamma v_i \nabla_j v_k+l_k k_{ji},\end{aligned} \tag{12}$$

where $\alpha_k=\partial_k\alpha$, etc.

Transvecting $g^{ji}$ to (12) gives an expression for $l^t k_{kt}$, where $l^t=g^{ts}l_s$. Making use of this expression for $l^t k_{kt}$ in the transvection of $l^j$ to (12) gives the following formula for $l^2 k_{ki}$, where $l^2=l^t l_t$.

$$\begin{aligned}l^2 k_{ki}=&(l\alpha)g_{ki}+(l\beta)u_k u_i+(l\gamma)v_k v_i\\&+k_t{}^t l_k l_i+[(\beta u)+\beta\nabla^t u_t]l_k u_i+[(\gamma v)+\gamma\nabla^t v_t]l_k v_i\\&-(n-1)l_k\alpha_i-\alpha_k l_i-l_k\beta_i-l_k\gamma_i-(lu)\beta_k u_i-(lv)\gamma_k v_i\end{aligned} \tag{13}$$

$$+\beta(u_k l^t+l_k u^t)\nabla_t u_i+\gamma(v_k l^t+l_k v^t)\nabla_t v_i+\beta u_i l^t(\nabla_t u_k-\nabla_k u_t)$$
$$+\gamma v_i l^t(\nabla_t v_k-\nabla_k v_t)-\beta(lu)\nabla_k u_i-\gamma(lv)\nabla_k v_i,$$

where $(l\alpha)=l^t\alpha_t$, $\nabla^t=g^{ts}\nabla_s$, $k_t{}^t=k_{ts}g^{st}$, etc.

On the other hand, by transvecting $u^j$ to (12) and by transvecting $u^i$ to the resulting formula, we obtain the following.

$$
\begin{aligned}
(lu)^2k_{ki}=&(lu)(\alpha u)g_{ki}+(lu)(\beta u)u_k u_i+(lu)(\gamma u)v_k v_i\\
&+k(u,u)l_k l_i+[(\alpha u)+(\beta u)]l_k u_i+(\gamma u^s u^t\nabla_s v_t)l_k v_i\\
&-l_k\alpha_i-l_k\beta_i-(lu)\beta_k u_i-(lu)\alpha_k u_i+\beta l_k u^t\nabla_t u_i\\
&+\beta(lu)(u_i u^t\nabla_t u_k+u_k u^t\nabla_t u_i)+\gamma(lu)(v_i u^t\nabla_t v_k+v_k u^t\nabla_t v_i)\\
&-\beta(lu)\nabla_k u_i-\gamma(lu)v_i u^t\nabla_k v_t,
\end{aligned}
\tag{14}
$$

where $k(u,u)=k_{ts}u^t u^s$. Analogous calculations starting from (12) with $v^k$ instead of $u^k$, give an analogous formula for $(lv)^2k_{ki}$.

From these formulas for $l^2k_{ki}$, $(lu)^2k_{ki}$ and $(lv)^2k_{ki}$ we get

$$
\begin{aligned}
&[l^2-(lu)^2-(lv)^2]k_{ki}\\
&=[(l\alpha)-(lu)(\alpha u)-(lv)(\alpha v)]g_{ki}+[(l\beta)-(lu)(\beta u)-(lv)(\beta v)]u_k u_i\\
&\quad+[(l\gamma)-(lu)(\gamma u)-(lv)(\gamma v)]v_k v_i+[k_t{}^t-k(u,u)-k(v,v)]l_k l_i\\
&\quad+[\beta\nabla^t u_t-(\alpha u)-\beta v^s v^t\nabla_s u_t]l_k u_i+[\gamma\nabla^t v_t-(\alpha v)-\gamma u^s u^t\nabla_s v_t]l_k v_i\\
&\quad-(n-3)l_k\alpha_i-\alpha_k l_i+(lu)\alpha_k u_i+(lv)\alpha_k v_i\\
&\quad+\beta u_k l^t\nabla_t u_i+\gamma v_k l^t\nabla_t u_i+\beta u_i l^t(\nabla_t u_k-\nabla_k u_t)\\
&\quad+\gamma v_i l^t(\nabla_t v_k-\nabla_k v_t)-\beta(lu)(u_i u^t\nabla_t u_k+u_k u^t\nabla_t u_i)\\
&\quad-\gamma(lu)(v_i u^t\nabla_t v_k+v_k u^t\nabla_t v_i)+\gamma(lu)v_i u^t\nabla_k v_t-\beta(lv)(u_i v^t\nabla_t u_k+u_k v^t\nabla_t u_i)\\
&\quad+\beta(lv)u_i v^t\nabla_k u_t-\gamma(lv)(v_i v^t\nabla_t v_k+v_k v^t\nabla_t v_i).
\end{aligned}
\tag{15}
$$

Making use of formula (15) and the symmetry of $k_{ki}$, one can derive expressions for $\beta l^t\nabla_k u_t$ and $\gamma l^t\nabla_k u_t$. Inserting these expressions in (15) and then again making use of the symmetry of $k_{ki}$, one can find an expression for $\alpha_k$. Finally, substituting this one in the previous formula, gives the following.

LEMMA 1. *Let $h_{ki}=\alpha g_{ki}+\beta u_k u_i+\gamma v_k v_i$. If $D_j\xi\equiv l_j\xi^\perp\neq 0$, $l_j\notin$ plane spanned by $u_j$ and $v_j$, $(lu)\equiv l_i u^i\neq 0$ and $(lv)\equiv l_i v^i\neq 0$ at a point $p$, then at $p$ we have*

$$
\begin{aligned}
h_{ki}=&[l^2-(lu)^2-(lv)^2]^{-1}[[(l\alpha)-(lu)(\alpha u)-(lv)(\alpha v)]g_{ki}\\
&+\{(l\beta)-(lu)(\beta u)-(lv)(\beta v)+\beta(lu)\nabla^t u_t-\beta(lu)v^s v^t\nabla_s u_t+\beta(lv)u^s v^t\nabla_s u_t\\
&-\beta u^s l^t\nabla_s u_t-(n-4)(lu)[l^2-(lu)^2-(lv)^2]^{-1}[l^2(\alpha u)+(lu)(lv)(\alpha v)
\end{aligned}
$$

$$-(lu)(l\alpha)-(\alpha u)(lv)^2]\}u_ku_i+\{(l\gamma)-(lu)(\gamma u)-(lv)(\gamma v)+\gamma(lv)\nabla^t v_t$$
$$-\gamma(lv)u^s u^t\nabla_s v_t+\gamma(lu)v^s v^t\nabla_s v_t-\gamma v^s l^t\nabla_s v_t$$
$$-(n-4)(lv)[l^2-(lu)^2-(lv)^2]^{-1}[l^2(\alpha v)+(lu)(lv)(\alpha u)-(lv)(l\alpha)$$
$$-(\alpha v)(lu)^2]\}v_kv_i+\{k_t{}^t-k(u,u)-k(v,v)-(n-2)[l^2-(lu)^2-(lv)^2]^{-1}$$
$$[(l\alpha)-(\alpha u)(lu)-(\alpha v)(lv)]\}l_kl_i+[l^2-(lu)^2-(lv)^2]^{-1}[-l^2(\alpha u)$$
$$+(\alpha u)(lv)^2+(lu)(l\alpha)-(lu)(\alpha v)(lv)](l_ku_i+l_iu_k)$$
$$+[l^2-(lu)^2-(lv)^2]^{-1}[-l^2(\alpha v)+(\alpha v)(lu)^2+(lv)(l\alpha)$$
$$-(lu)(lv)(\alpha u)](l_kv_i+l_kv_i)+\frac{1}{2}\{\beta(lv)\nabla^t u_t-(\alpha u)(lv)+(lu)(\alpha v)$$
$$-\beta v^s l^t\nabla_s u_t-(n-4)(lv)[l^2-(lu)^2-(lv)^2]^{-1}[l^2(\alpha u)+(lu)(lv)(\alpha v)$$
$$-(lu)(l\alpha)-(lu)(lv)^2]+\gamma(lu)\nabla^t v_t-(\alpha v)(lu)+(lv)(\alpha u)-\gamma u^s l^t\nabla_s v_t$$
$$-(n-4)(lu)[l^2-(lu)^2-(lv)^2]^{-1}[l^2(\alpha v)+(lu)(lv)(\alpha u)-(lv)(l\alpha)$$
$$-(\alpha v)(lu)^2]\}(v_iu_k+v_ku_i)+\beta[l^2-(lu)^2-(lv)^2]^{-1}\{u_k[l^t-(lu)u^t$$
$$-(lv)v^t]\nabla_t u_i+u_i[l^t-(lu)u^t-(lv)v^t]\nabla_t u_k\}$$
$$+\gamma[l^2-(lu)^2-(lv)^2]^{-1}\{v_k[l^t-(lu)u^t-(lv)v^t]\nabla_t v_i$$
$$+v_i[l^t-(lu)u^t-(lv)v^t]\nabla_t v_k\}].$$

From Lemma 1 and the expressions one can obtain for $k_{ki}$ in a similar way in case $l_j\notin$ plane $(u_j, v_j)$ with $l_j\perp u_j$ or $l_j\perp v_j$, we have the following.

THEOREM 2. *Let* $h_{ji}=\alpha g_{ji}+\beta u_ju_i+\gamma v_jv_i$. *If* $D_j\xi\equiv l_j\xi^\perp\neq 0$ *and* $l_j\notin$ *plane spanned by* $u_j$ *and* $v_j$ *at a point* $p$, *then at* $p$ $k_{ji}$ *has an eigenvalue with multiplicity* $\geqq n-5$.

REMARK 1. More generally we have the following: if $\xi$ is a $q$-quasi-umbilical non-parallel normal section of an $n$-dimensional submanifold $M$ with codimension 2 in a space form and the third fundamental tensor does not belong to the $q$-plane spanned by the distinguished directions of $M$ with respect to $\xi$ at a point $p$, then at $p$ $A_{\xi^\perp}$ has an eigenvalue with multiplicity $\geqq n-2q-1$.

Also by similar calculations, we obtain the following.

LEMMA 3. *Let* $h_{ki}=\alpha g_{ki}+\beta u_ku_i+\gamma v_kv_i$.

(i) *If* $D_j\xi\equiv l_j\xi^\perp\neq 0$, $l_j\in$ *plane spanned by* $u_j$ *and* $v_j$, *and* $l_j\nparallel u_j$ *and* $l_j\nparallel v_j$ *at a point* $p$, *then at* $p$ *we have*

$$k_{ki}=l^{-2}\{(l\alpha)g_{ki}+[(l\beta)+k_t{}^t(lu)^2+(\beta u)(lu)+\beta(lu)\nabla^t u_t-n(lu)(\alpha u)]u_ku_i$$
$$+[(l\gamma)+k_t{}^t(lv)^2+(\gamma v)(lv)+\gamma(lv)\nabla^t v_t-n(lv)(\alpha v)]v_kv_i$$
$$+\frac{1}{2}[2k_t{}^t(lu)(lv)+(lu((\gamma v)+\gamma(lu)\nabla^t v_t$$
$$-(n-1)(lu)(\alpha v)-(\alpha u)(lv)+(\beta u)(lv)+\beta(lv)\nabla^t u_t$$
$$-(n-1)(lv)(\alpha u)-(\alpha v)(lu)](u_kv_i+u_iv_k)$$
$$-(lu)(\beta_ku_i+u_k\beta_i))-(lv)(\gamma_kv_i+\gamma_iv_k)-(lv)\beta_iv_k-(lu)\gamma_iu_k$$
$$-\beta(lu)\nabla_kv_i-\gamma(lv)\nabla_kv_i+\beta(lu)(u_iu^t\nabla_tu_k+2u_ku^t\nabla_tu_i)$$
$$+\gamma(lv)(v_iv^t\nabla_tv_k+2v_kv^t\nabla_tv_i)+\beta(lv)(u_kv^t\nabla_tu_i+u_iv^t\nabla_tu_k)$$
$$+\gamma(lu)(v_ku^t\nabla_tv_i+v_iu^t\nabla_t+v_iu^t\nabla_tv_k)+\beta(lv)(v_ku^t\nabla_tu_i-u_iv^t\nabla_kv_t)$$
$$+\gamma(lu)(u_kv^t\nabla_tv_i-v_iu^t\nabla_kv_t).$$

(ii) *If* $D_j\xi\equiv l_j\xi^\perp\neq 0$ *and* $l_j/\!/u_j$ *at a point* $p$, *then at* $p$ *we have*

$$k_{ki}=(\alpha u)g_{ki}+[2(\beta u)+k_t{}^t+\beta\nabla^t u_t-n(\alpha u)]u_ku_i+(\gamma u)v_kv_i+\frac{1}{2}[(\gamma v)$$
$$+\gamma\nabla^t v_t-n(\alpha v)]\ (u_kv_i+u_iv_k)-u_k\beta_i-\beta_ku_i-\beta\nabla_ku_i-\gamma_iu_k+\beta(u_iu^t\nabla_tu_k$$
$$+2u_ku^t\nabla_tu_i)+\gamma(v_ku^t\nabla_tv_i+v_iu^t\nabla_tv_k)+\gamma(u_kv^t\nabla_tv_i-u_iu^t\nabla_kv_t).$$

## §4. Foliations of submanifolds with a 2-quasi-umbilical normal direction

In this paragrapf we give results on foliations of submanifolds $M^n$ of $M^{n+2}(c)$ which are 2-quasi-umbilical w.r.t. a normal section $\xi$.

Let $l_j\neq 0$, $l_j\notin$ plane $(u_j, v_j)$, $(lu)\neq 0$ and $(lv)\neq 0$. Then from formula (10) and Lemma 1, the equation (8) of Ricci and the formulas obtained by transvecting $u^j$, respectively $v^j$, to (12), we see that $\nabla_jl_i-\nabla_il_j$, $\nabla_ju_i-\nabla_iu_j$ and $\nabla_jv_i-\nabla_iv_j$ are all zero modulo $u_k$, $v_k$ and $l_k$. Therefore, the distribution defined by $u_jdx^j=v_jdx^j=l_jdx^j=0$ is involutive. From (10) we see that the corresponding integral submanifolds $M^{n-3}$ of $M^n$ are umbilical w.r.t. the normal direction $\xi$. From Lemma 1 we see that also $\xi^\perp$ determines an umbilical normal direction for $M^{n-3}$. Finally from the transvection of $u^j$, respectively $v^j$, to (12) we see that also $u_j$ and $v_j$ determine umbilical normal directions for $M^{n-3}$. Besides, from (2) and (10) we see that

$$D'_{X'}\xi=D_{X'}\xi \tag{16}$$

where $D'$ and $X'$ are respectively the normal connection of $M^{n-3}$ in $M^{n+2}(c)$

and an arbitrary tangent vector field on $M^{n-3}$. (16) and (4) imply that $\xi$ is parallel on $M^{n-3}$.

The following result can be proved completely by making similar observations for the other cases as given above for case (i). In particular for the cases (ii) and (iii) one uses Lemma 3. Of course in case (iv), the parallellism of $\xi$ on the integral submanifolds is trivial.

THEOREM 4. *Let $M^n$ be a submanifold of $M^{n+2}(c)$ with*

$$h_{ji}=\alpha g_{ji}+\beta u_j u_i+\gamma v_j v_i.$$

(i) *If $l_j\neq 0$ and $l_j\notin$ plane spanned by $u_j$ and $v_j$, then $M^n$ is foliated by submanifolds of codimension* 3 *which are umbilical* w. r. t. $\xi$, $\xi^\perp$, $u_j$ *and* $v_j$.

(ii) *If $l_j\neq 0$, $l_j\in$ plane spanned by $u_j$ and $v_j$, $l_j \nparallel u_j$ and $l_j \nparallel v_j$, then $M^n$ is foliated by submanifolds of codimension* 2 *which are umbilical* w. r. t. $\xi$ *and two independent normal directions spanned by* $\xi^\perp$, $u_j$ and $v_j$

(iii) *If $l_j\neq 0$ and $l_j /\!/ u_j$* (resp. $l_j /\!/ v_j$), *then $M^n$ is foliated by submanifolds of codimension* 2 *which are umbilical* w. r. t. $\xi$, $v_j$ (resp. $u_j$) *and a normal direction spanned by $\xi^\perp$ and $u_j$* (resp. $\xi^\perp$ *and* $v_j$).

(iv) *If $l_j=0$, then $M^n$ is foliated by submanifolds of codimension* 2 *which are umbilical* w. r. t. $\xi$, $u_j$ *and* $v_j$.

*In each case $\xi$ is parallel in the normal bundle of the integral submanifolds of $M^n$ in $M^{n+2}$.*

REMARK 2. On submanifolds $M^n$ of $M^{n+2}$ with a $q$-quasi-umbilical direction there also exist foliations of which the leaves $M^d$ are umbilical w. r. t. an $(n+1-d)$-dimensional normal subbundle.

REMARK 3. From Theorem 4 and [2] we conclude that the leaves $M^{n-2}(M^{n-3})$ of $M^n$ are spherical submanifolds, i. e. lie in a hypersphere of $M^{n+2}(c)$.

REMARK 4. For results on submanifolds $M^n$ of a Riemannian manifold $N^m$ which are umbilical w. r. t. an $(m-n-1)$-dimensional normal subbundle, see [3] and [4].

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