J. Korean Math. Soc.
Vol. 15, No. 2, 1979

IDEAL THEORY OF ORDERED LOCALLY CONVEX SPACES(*)

BY BYUNG SOO MOON

1. Introduction

In an ordered locally convex space, there are many special types of subspaces such as order ideals, positively generated subspaces and order direct summands. Other than these well known examples, subspaces with special order properties are: perfect subspaces, full ideals, subspaces with Property (I_1), subspaces with Property (I_2) and strict ideals.

Perfect subspaces are introduced and studied in a paper by Ellis [3]. It is proved there that a closed order ideal I of a partially ordered vector space with order unit is perfect if and only if I^0 is an order ideal in the dual space. We see that in [5], Nagel extends the definition of a perfect subspace to any ordered locally convex space. Full ideals and fully perfect ideals are also introduced in [5] and it is proved that a subspace of a locally convex vector lattice is a fully perfect ideal if and only if it is a closed lattice ideal.

Strict ideals, subspaces with Property (I_1) and subspaces with Property (I_2) are studied in [2]. Strict ideals are found to have more order properties than fully perfect ideals and yet these concepts agree in locally convex lattices. Asimow [1] considers subspaces which have nearly the same order properties as strict ideals.

The objective of this paper is to study the inter-relations among these order properties and to establish the duality between certain of these properties.

A schematic diagram shown at the end of this paper illustrates the inter-relations among the order properties and the duality relations that we have studied.

Throughout this paper, unless otherwise stated, E denotes an ordered locally convex space (OLCS) over the real field $\mathbf{R}$, with a positive cone C which is always assumed to be closed. If I is a subspace then $\hat{x}$ will denote the image of x under the canonical map $\phi: E \rightarrow E/I$. In quotient spaces, the cone is not assumed to be closed, nor proper. If M is a subset of E, $[M]$ is

Received Oct. 1, 1978

(*) This is a part of the Ph. D. Thesis submitted to University of Illinois, 1974.

the C-saturated hull of M, $(M+C)\cap(M-C)$. The convex hull of a subset M in E is denoted by $\Gamma(M)$ and the closure of the convex hull is denoted by $\bar{\Gamma}(M)$.

(1.1) DEFINITION. Let E be an OLCS.

(a) A subspace I of E is *perfect* if for each 0-neighborhood U in E and for every $a\in I$, there exist $b\in I$, $u\in U$ and $v\in U$ such that $-u-b\leq a\leq b+v$ (Ellis [3]).

(b) A subspace I is *positively generated* if for every $a\in I$, there exists a positive element b of I such that $a\leq b$.

(c) A subspace I has *Property* (I_1') if for every positive element x of E and $a\in I$ with $a\leq x$ and for every 0-neighborhood U in E, there exist a positive element b in I and $u\in U$ such that $a\leq b\leq x+u$ (Combes and Perdrizet [2]).

(d) A subspace I has *Property* (I_1) if for every $a\in I$ and positive element x of E with $a\leq x$, there exists a positive element b of E such that $a\leq b\leq x$ (Combes and Perdrizet [2]).

We note that the following implications hold: (d) $\Longrightarrow$ (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a). Note also that the Property (I_1) corresponds to the order property of a sublattice in a vector lattice.

(1.2) DEFINITION. Let E be an OLCS and let I be a subspace of E.

(a) The subspace I is an *order ideal* if for every $a\in I$, the order interval $[0,a]$ is contained in I.

(b) The subspace I is an *ideal* if it is a positively generated order ideal.

(c) The subspace I is a *full ideal* if $I=\bigcap_{u\in\mathcal{U}}[I+U]$, where $\mathcal{U}$ is a 0-neighborhood base in E (Nagel [5]).

(d) The subspace I is a *fully perfect ideal* (*fp-ideal*) if $I=\{x\in E\mid \forall U\in\mathcal{U},\ \exists u,v\in U, y\in I \text{ such that } -u-y\leq x\leq v+y\}$ (Nagel [5]).

(e) The subspace I has *Property* (N) if for every 0-neighborhood U in E, there is a 0-neighborhood V such that $[(I+V)\cap C]\subseteq I+U$ (cf. Jameson [4]).

(f) The subspace I has *Property* (I_2') if for all $a\in I$ and all positive elements x and y of E with $a\leq x+y$ and for every 0-neighborhood U in E, there exist b and c in I and $u\in U$ such that $b\leq x$, $c\leq y$ and $a\leq b+c+u$ (Combes and Perdrizet [2]).

(g) The subspace I has *Property* (I_2) if for all $a\in I$, positive elements x and y of E with $a\leq x+y$, there exist b and c in I such that $b\leq x$, $c\leq y$ and $a\leq b+c$ (Combes and Perdrizet [2]).

(h) The subspace I is a *strict ideal* if I is an order ideal and if I has both Properties (I_1') and (I_2') (Combes and Perdrizet [2]).

In the definition of Properties (I_1') and (I_2'), Combes and Perdrizet [2] take U to be a $\sigma(E, E')$ 0-neighborhood. There, a *strict ideal* in an ordered vector space is an order ideal with Properties (I_1) and (I_2). A *nearly strict ideal* in an ordered normed space is an order ideal with Properties (I_1') and (I_2). Fully perfect ideals correspond to closed lattice ideals in a locally convex lattice in the sense that a subspace of a locally convex lattice is a fully perfect ideal if and only if it is a closed lattice ideal.

It is also easy to see that in a Banach lattice, a closed subspace is a strict ideal if and only if it is a closed lattice ideal, and the strict ideals have stronger order properties than fully perfect ideals in general. An order ideal I of an ordered topological vector space E with a positive cone C has Property (N) if and only if $\hat{C}$ is a normal cone in E/I.

2. Order ideals with Property (I_2') or (I_2).

In this section, we consider an equivalent condition for the Property (I_2') and prove that in an OLCS with a normal cone that gives an open decomposition, if I is an order ideal with Property (I_2'), then I^0 has Property (I_1). The converse of this and a further study is given in section 4.

(2.1) LEMMA. *Let E be an OLCS with a normal cone that gives an open decomposition and let I be a subspace. Then the following are equivalent:*

(1) *The subspace I has Property* (I_2').

(2) *If x is positive and $\hat{0} \leq \hat{z} \leq \hat{x}$, then for each 0-neighborhood U in E, there exist $a \in I$ and $u \in U$ such that $0 \leq z+a \leq x+u$.*

Proof. (1) $\Longrightarrow$ (2). Let $\hat{0} \leq \hat{z} \leq \hat{x}$, where x is a positive element of E. Then $0 \leq z+b \leq x+a$ for some a and b in I. If $y=x+a-(z+b)$ and $w=z+b$, then y and w are positive. Since $a \leq y+w$, for each 0-neighborhood U in E, there exist $c \in I$, $d \in I$ and $u \in U$ such that $c \leq y$, $d \leq w$ and $a \leq c+d+u$. Hence, $c \leq x+a-(z+b)$, $d \leq z+b$ and $a \leq c+d+u$. From these, we obtain $0 \leq z+b \leq x+a-c$ and $a-c \leq d+u$. Therefore, $0 \leq z+b-d \leq x+u$ with $b-d \in I$.

(2) $\Longrightarrow$ (1). Let $a \in I$ and let x and y be positive elements of E with $0 \leq x+y+a$. If U is a 0-neighborhood, then $V=U \cap C - U \cap C$ is a 0-neighborhood. From $0 \leq x+y+a$, we have $\hat{0} \leq \hat{x} \leq \hat{x}+\hat{y}+a$. By hypothesis, there exist $b \in I$ and $u \in \frac{1}{2}V$ such that $0 \leq x+b \leq x+y+a+u$. Without loss of generality, we may assume that u is positive. Then $\hat{0} \leq \hat{y} \leq \hat{y}+\hat{a}+\hat{u}-\hat{b}$ and by hypothesis again, $0 \leq y+c \leq y+a+u-b+v$ for some $c \in I$ and $v \in \frac{1}{2}V \cap C$. Thus we have $x+b$ and $y+c$ are positive, $b+c \leq a+u+v$ and $u+v \in U$.

In the above proof, it is clear that (1) implies (2) in any OLCS. Also we can prove similarly that in any ordered vector space, the Property (I_2) is equivalent to the other property that for every positive element x of E, $\hat{0}\leq\hat{z}\leq\hat{x}$ implies that there exists $a\in I$ such that $0\leq z+a\leq x$.

(2.2) PROPOSITION. *Let E be an* OLCS *with a normal cone that gives an open decomposition.*
If I is a subspace with Property (I_2'), *then I has Property* (N).

Proof. This follows easily from (2.1).

(2.3) COROLLARY. *Let E be an* OLCS *with a normal cone that gives an open decomposition. Then every closed subspace of E with Property* (I_2') *is an order ideal.*

Proof. This follows directly from (2.2) since a closed subspace with Property (N) is an order ideal.

In the following, we show that for a subspace I with Property (I_2'), there is a map γ from C' to $I^0\cap C'$. We make use of this map to prove that I^0 has the Property (I_1) (see (2.6)). In (3.4), we find that this map is a "projection" in the sense of (3.3) when the subspace I has also the Property (I_1').

(2.4) LEMMA. *Let E be an* OLCS *with a normal cone that gives an open decomposition and let I be a subspace with Property* (I_2'). *Then for each $f\in C'$,*

$$g(x)=\operatorname{Inf}\{f(z)\,|\,z\in(x+I)\cap C\}\quad(0\leq x\in E)$$

defines a linear form in $I^0\cap C'$.

Proof. Clearly we have $g(\lambda x)=\lambda g(x)$ for every positive x and every positive scalar λ. Let x and y be positive elements of E. It is easy to see that $g(x+y)\leq g(x)+g(y)$. We shall now show that $g(x)+g(y)\leq g(x+y)$. If ϵ is an arbitrary positive number, then there exists $a\in I$ such that $0\leq x+y+a$ and $f(x+y+a)\leq g(x+y)+\epsilon$. Let U be a 0-neighborhood in E such that $|f(U)|<\epsilon$. Then by Property (I_2'), there exist $b\in I$, $c\in I$ and $u\in U$ such that $0\leq x+b$, $0\leq y+c$ and $b+c\leq a+u$. Thus,

$$g(x)+g(y)\leq f(x+b)+f(y+c)\leq f(x+y+a+u)\leq g(x+y)+2\epsilon,$$

so that $g(x)+g(y)\leq g(x+y)$ for all x and y in C. Extend g to a linear form on E by defining $g(c-d)=g(c)-g(d)$ for all c and d in C. Clearly, g is linear and if $V=U\cap C-U\cap C$, where U is the same 0-neighborhood as above, then $|g(V)|<\epsilon$. Therefore, g is continuous and positive. It remains to show that $g\in I^0$. If $a\in I$, then $a=b-c$, where b and c are positive elements of E. Hence $b+I=c+I$ and $(b+I)\cap C=(c+I)\cap C$. Thus, $g(b)=g(c)$ and $g(a)=0$.

If for each $f \in C'$, we define $\gamma(f)=g$ where g is the functional associated with f as in (2.4), then we have a map γ from C' onto $I^0 \cap C'$ whenever I is a subspace with Property (I_2'). Note that $\gamma(f)=f$ for all f in $I^0 \cap C'$ and that for any f and g in C' with $f \leq g$, we have $\gamma(f) \leq \gamma(g)$.

(2.5) LEMMA. *Let E be an* OLCS *with a normal cone that gives an open decomposition and let I be an order ideal of E with Property* (I_2').
If $f \in C'$ and $g \in I^0$ with $g \leq f$, then $g \leq \gamma(f)$.

Proof. By (2.2), the subspace I has Property (N), hence by II. 1. 21 in [6], I^0 is positively generated. If $g=h-k$, where h and k are positive elements of I^0, we have $h \leq f+k$ and so by the above remark, $h \leq \gamma(f+k)$. To conclude the proof, we only have to note that $\gamma(f+k)=\gamma(f)+k$ since $k \in I^0$.

(2.6) THEOREM. *Let E be an* OLCS *with a normal cone that gives an open decomposition and let I be an order ideal with Property* (I_2') *Then I^0 has Property* (I_1).

Proof. Let g and h be in I^0. If f is in C' and if $f+g$ and $f+h$ are positive, then $-g \leq f$ and $-h \leq f$. Now, by (2.5), $-g \leq \gamma(f)$ and $-h \leq \gamma(f)$. Therefore $0 \leq f-\gamma(f) \leq f+g$ and $f-\gamma(f) \leq f+h$.

(2.7) EXAMPLE. Let E be an OLCS and H be a maximal closed order ideal. If $\hat{0} \leq \hat{y} \leq \hat{x}$, where x is positive, then $y=\lambda x+h$ for some $\lambda \in \mathbf{R}$ and $h \in H$. Hence from $\hat{0} \leq \lambda \hat{x} \leq \hat{x}$, we have $0 \leq \lambda \leq 1$. Therefore $0 \leq \lambda x \leq x$, and $\lambda \hat{x}=\hat{y}$, which shows that H has Property (I_2).

3. Subspaces with Property (I_1') and strict ideals.

In this section, we prove that in an OLCS with a normal cone that gives an open decomposition, if a subspace I of E has Property (I_1'), then I^0 has Property (I_2), and if I is a strict ideal then $I^0 \cap C'$ is complemented in C' (see (3.3) below for the definition).

(3.1) LEMMA. *Let E be an* OLCS *with a positive cone C that gives an open decomposition and let I be a subspace of E with Property* (I_1'). *If h is a continuous positive linear form on E and g is a continuous positive linear form on I such that $g \leq h|_I$, then g has a continuous positive linear extension f on E such that $f \leq h$.*

Proof. We define a map δ from E to $\mathbf{R}$ such that

$$\delta(x)=\operatorname{Inf}\{h(z) \mid x \leq z,\ 0 \leq z\}.$$

Then it is easy to check that δ is a seminorm on E. We claim that $g(a) \leq$

$\delta(a)$ for all $a\in I$. To see this, let z be positive with $a\leq z$ and let ϵ be an arbitrary positive number. Then by Property (I_1'), there exist a positive element b in I and $u\in U$ such that $a\leq b\leq z+u$, where we take U to be the 0-neighborhood $\{x\in E \mid |h(x)|<\epsilon\}$. Hence, $h(b)\leq h(z+u)\leq h(z)+\epsilon$ and $g(a)=g(b)-g(b-a)\leq g(b)\leq h(b)$. From these, we conclude $g(a)\leq\delta(a)$.

Now, by the Hahn-Banach Theorem, extend g to f so that $f(x)\leq\delta(x)$ for all $x\in E$. The extension f is positive since for each positive c, we have $f(-c)\leq\delta(-c)=0$. And $f\leq h$, from $f(c)\leq\delta(c)=h(c)$. Furthermore, if $V=U\cap C-U\cap C$, where U is the 0-neighborhood given above, then we have $|f(V)|<\epsilon$. Therefore, f is continuous.

(3.2) PROPOSITION. *Let E be an* OLCS *with a positive cone C that gives an open decomposition. If I is a subspace of E with Property* (I_1'), *then I^0 has Property* (I_2).

Proof. Let $f\in C'$, $g\in C'$ and let $k\in I^0$ such that $0\leq f+g+k$. Then $0\leq g|_I\leq(f+g+k)|_I$. Hence, by (3.1) there exists $h\in I^0$ such that $g+h$ is positive and $g+h\leq f+g+k$. Therefore, $f+(k-h)$ and $g+h$ are positive with sum $f+g+k$.

(3.3) DEFINITION. Let E be an ordered vector space with a positive cone C. Then a subcone (wedge) K of C is *complemented* in C if there exists a map γ from C onto K such that;

(1) $\gamma(x+y)=\gamma(x)+\gamma(y)$

(2) $\gamma(\lambda x)=\lambda\gamma(x)$, and

(3) $\gamma^2(x)=\gamma(x)\leq x$

for all $\lambda\geq 0$, $x\in C$ and $y\in C$.

(3.4) THEOREM. *Let E be an* OLCS *with a normal cone that gives an open decomposition. If I is a closed strict ideal of E, then $I^0\cap C'$ is complemented in C'.*

Proof. In (2.5) we have defined a map γ from C' onto $I^0\cap C'$. We show in the following that γ satisfies the conditions (1), (2) and (3) in (3.3). Recall that $\gamma(f)\leq f$ for all $f\in C'$ and $\gamma(f)=f$ if $f\in I^0\cap C'$. Hence (3) is trivial to verify and the routine check for (2) is omitted. To prove (1), let f and g be in C'. From the definition of γ it follows that $\gamma(f)+\gamma(g)\leq\gamma(f+g)$. Let x be any positive element of E and ϵ be an arbitrary positive number. Then there exists $a\in I$ such that $x+a$ is positive and $f(x+a)\leq\gamma(f)(x)+\epsilon$. Similarly, $g(x+b)\leq\gamma(g)(x)+\epsilon$ for some $b\in I$ with $x+b\geq 0$. By Property (I_1'), if U is a 0-neighborhood with $|f(U)|<\epsilon$ and $|g(U)|<\epsilon$, then there exist $c\in I$, $u\in U$ and $v\in U$ such that $0\leq x+c\leq x+a+u$ and $x+c\leq x+b+v$. Consequently,

$$\gamma(f+g)(x)\le(f+g)(x+c)\le f(x+a+u)+g(x+b+v)$$
$$\le\gamma(f)(x)+\gamma(g)(x)+4\epsilon.$$

From this, we conclude $\gamma(f+g)\le\gamma(f)+\gamma(g)$.

(3.5) PROPOSITION. *Let E be an OLCS with a normal cone that gives an open decomposition and I be a subspace of E. Then the following are equivalent:*

(a) *The subspace I is a strict ideal.*

(b) *For any positive x and y, if $\hat{0}\le\hat{w}\le\hat{x}$ and $\hat{w}\le\hat{y}$, then for each 0-neighborhood U in E, there exist $a\in I$ and $u\in U$ such that $0\le w+a\le x+u$ and $w+a\le y+u$.*

The routine proof of this is omitted.

(3.6) DEFINITION. An ordered normed space E is *regularly ordered* if:

(a) for each x in E and every positive number ϵ, there exists y in E with $-y\le x\le y$ and $\|y\|\le\|x\|+\epsilon$,

(b) for every x and y with $-x\le y\le x$, $\|y\|\le\|x\|$.

A *GM*-space is a regularly ordered Banach space such that for any x and y in the open unit ball, there exists z in the open unit ball with $x\le z$ and $y\le z$.

A *GL*-space is a regularly ordered Banach space such that the norm is additive on the positive cone.

Note that if E is a regularly ordered normed space, then clearly the positive cone is normal and gives an open decomposition. Hence, we have the following corollaries to (3.4)

(3.7) COROLLARY. *Let E be a regularly ordered Banach space and let I be a closed strict ideal of E. Then the quotient cone in E/I is norm-closed.*

Proof. This follows from (3.4) and Proposition (3.12) in [5].

(3.8) COROLLARY. *Let E be a GL-space and let I be a closed strict ideal in E. Then $I\cap C$ is complemented in C.*

Proof. This also follows from (3.4) and Proposition (4.11) in [5].

(3.9) EXAMPLE. (1). Let E be a positively generated ordered vector space with the Riesz decomposition property. Then every positively generated order ideal in E satisfies both of the Properties (I_1) and (I_2), and hence it is a strict ideal.

Proof. To show that I has Property (I_1), let z be positive and $a\in I$ with $a\le z$. Then, since I is positively generated, we have a positive element b of I such that $a\le b$. Now, by the Riesz decomposition property (or the interpolation property), there exists $w\in E$ such that

$$a, 0 \leq w \leq b, z.$$

Since I is an order ideal, it is clear that w is in I.

To prove that I has Property (I_2), let x be positive and $\hat{0} \leq \hat{z} \leq \hat{x}$. Then for some a and b in I, we have $0 \leq z+a \leq x+b$. We may assume that b is positive so that we have

$$a-b, \ -z \leq x-z, \ a.$$

Now, by the decomposition property, there exists $w \in E$ such that

$$a-b, \ -z \leq w \leq x-z, \ a.$$

Then $w \in I$ and $0 \leq z+w \leq x$.

EXAMPLE (2). As an example of a strict ideal in an OLCS which does not have the Riesz decomposition property, we can take E to be the Hermitian part of a C^*-algebra with identity and I to be the Hermitian part of a norm-closed two-sided ideal (see Proposition (3.4) in [2]).

(3.10) DEFINITION. Let E be an ordered vector space with positive cone C and let F be a convex subset of C. Then F is called a *Face* of C if all the positive x and y are contained in F, whenever $\lambda x+(1-\lambda)y \in F$ for some $0<\lambda<1$.

A face F is called a *Split Face* if there exists a face G of C such that $(F-F) \cap (G-G)=0$ and $F+G=C$.

Note that a face F of C is a split face if and only if F is complemented in C.

(3.11) PROPOSITION. *Let E be an* OLCS *with a normal cone that gives an open decomposition and let E_1 and E_2 be closed subspaces of E such that E is the order topological direct sum of E_1 and E_2. Then E_σ'* is the order topological direct sum of $E'_{1\sigma}$ and $E'_{2\sigma}$.

Proof. Note that E_1 and E_2 are positively generated. Also $E_1 \cap C$ and $E_2 \cap C$ are split faces of C. Hence E_1 has both Properties (I_1) and (I_2). Therefore by (3.4), $E_1^0 \cap C'$ is a split face of C' and E_1^0 can be identified with E_2'. Thus, E_2' is an order direct summand of E$'$.

In the previous sections, we proved that in an ordered locally convex space E which has a normal cone that gives an open decomposition, if I is a subspace with Property (I_1'), then I^0 has Property (I_2) and if I has Property (I_2'), then I^0 has Property (I_1). In the following, we define Properties (I_1'') and (I_2'') and prove that in any OLCS with a normal cone that gives an open decomposition, a closed subspace I has Property (I_1'') if and only if I^0 has Property (I_2), and I has Property (I_2'') if and only if I^0 has Property (I_1) (see (4.7) and (5.6)). Also, we find that the expected implications for closed subspaces:

$$(\mathrm{I}_1') \Longrightarrow (\mathrm{I}_1'') \Longrightarrow \text{perfect subspace, and}$$
$$(\mathrm{I}_2') \Longrightarrow (\mathrm{I}_2'') \Longrightarrow (\mathrm{N})$$

are true.

Finally, in section 6, it is proved that in a regularly ordered Banach space, if a closed subspace has both Properties (I_1'') and (I_2''), then it has Properties (I_1') and (I_2') (see (6.6)).

4. Property (I_2'').

(4.1) Definition. Let E be an OLCS and let I be a subspace of E. Then I has:

Property (I_1'') if for every positive $x \in E$ and $a \in I$ with $a \leq x$, and for every 0-neighborhood U in E, there exist $b \in I$ and $u \in U$ such that $0 \leq b \leq x$ and $a \leq b+u$.

Property (I_2'') if for every positive x with $\hat{0} \leq \hat{z} \leq \hat{x}$ and for every 0-neighborhood U in E, there exist $a \in I$ and $u \in U$ such that $0 \leq z+a+u \leq x$, where $\hat{x}$ is the image under the canonical map $\phi: E \rightarrow E/I$.

We will see later that a closed subspace with Property (I_2'') is an order ideal and any subspace with Property (I_1'') is necessarily a perfect subspace.

(4.2) Proposition. *Let E be an* OLCS *and let I be a closed subspace of E such that I^0 has Property* (I_1). *Then I has Property* (I_2'') *in* $\sigma(E, E')$.

Proof. Let $0 \leq x$ and $\hat{0} \leq \hat{z} \leq \hat{x}$. For every 0-neighborhood U in E, it is desired to find $a \in I$ and $u \in U$ such that $0 \leq z+a+u \leq x$. Note that if z satisfies $\hat{0} \leq \hat{z} \leq \hat{x}$, then $z \in (x+I-C) \cap (I+C)$ and hence that it is enough to prove: $(x+I-C) \cap (I+C) \subseteq \overline{I+(x-C) \cap C}$. We claim that

$$(*) \quad I^0 \cap \Gamma\{(\{x\}^0 \cap C') \cap (-C')\} \subseteq \Gamma\{(I^0 \cap C' \cap \{x\}^0) \cup (I^0 \cap (-C')\}.$$

Assume for a moment that the claim is true. Taking polars, we obtain,

$$(I^0 \cap C' \cap \{x\}^0)^0 \cap (I^0 \cap (-C'))^0 \subseteq \bar{\Gamma}(I \cup \{(\{x\}^0 \cap C')^0 \cap C\}).$$

Now observe that $(x+I-C) \subseteq (I^0 \cap C' \cap \{x\}^0)^0$, $I+C \subseteq (I^0 \cap -C')^0$ and $(\{x\}^0 \cap C')^0 = x-C$. Thus we finally have

$$(x+I-C) \cap (I+C) \subseteq \bar{\Gamma}(I \cup \{(\{x\}^0 \cap C')^0 \cap C\}) \subseteq \overline{I+(x-C) \cap C},$$

which is what was to be proved. To prove the claim, let f be an element of the left hand side of (*). Then $f = \lambda f_1 - (1-\lambda) f_2$, where f_1 and f_2 are positive, $f_1(x) \leq 1$ and $0 \leq \lambda \leq 1$. When $\lambda = 0$ we have nothing to prove. Hence assume $0 < \lambda$. Since I^0 has (I_1) and $f \leq \lambda f_1$, there exists $g \in I^0$ such that g is positive and $f \leq g \leq \lambda f_1$. Thus,

$$f = g-(g-f) = \lambda\{(1/\lambda) g\} - (1-\lambda)\{(1/(1-\lambda))(g-f)\}$$

with $g-f \in I^0 \cap C'$ and $(1/\lambda) g \in I^0 \cap C' \cap \{x\}^0$. Therefore f belongs to the set on the right of (*).

(4.3) COROLLARY. *Let E be an OLCS with normal cone that gives an open decomposition. Then a closed subspace with Property (I_2') satisfies Property (I_2'') in $\sigma(E, E')$.*

Proof. By (2.6), if a subspace I of E has Property (I_2'), then I^0 has Property (I_1). Hence by (4.2) above, I satisfies Property (I_2'').

(4.4) PROPOSITION. *Let E be an OLCS. If I is a subspace of E with Property (I_2''), then I^0 satisfies Property (I_1'') in $(E', \sigma(E', E))$.*

Proof. Let $g\in I^0$ and $f\in C'$ with $g\leq f$. We want to show that

$$g\in\overline{I^0\cap C'\cap(f-C')-C'},$$

which is sufficient for Property (I_1''). Let $U=\{x\in E \mid |g(x)|\leq 1\}$. We claim that

$$(*)\quad \Gamma\{(I-C)\cup(\{f\}^0\cap C\}\cap C\}\cap C\subseteq\Gamma\{(\{f\}^0\cap C)\cup I\cup U\}$$

Assume for a moment that the claim is true. We can take the polars of both sides to obtain,

$$\begin{aligned}U^0\cap I^0\cap(f-C')&\subseteq\bar{\Gamma}(\{(I-C)^0\cap(f-C')\}\cap(-C'))\\ &\subseteq\overline{I^0\cap C'\cap(f-C')-C'}.\end{aligned}$$

Now, we only have to note that $g\in U^0\cap I^0\cap(f-C')$ to get the conclusion. To prove the claim, let w be an element of the left hand side of (*). Then w is positive and $w=(1-\lambda)(a-z)+\lambda y$ for some positive z, $0\leq y\in\{f\}^0$, $a\in I$ and $0\leq\lambda\leq 1$. When $\lambda=1$, we have nothing to prove. Hence assume $\lambda<1$, then $\hat{0}\leq(1-\lambda)\hat{z}\leq\lambda\hat{y}$. Apply Property ($I_2''$) to find $b\in I$ and $v\in((1-\lambda)/2)U$ with $0\leq(1-\lambda)z+b+v\leq\lambda y$. Then $d=\lambda y-(1-\lambda)z-b-v$ is positive and $f(d)\leq\lambda f(y)\leq\lambda<1$. Moreover,

$$\begin{aligned}w&=(1-\lambda)(a-z)+\lambda y=d+(1-\lambda)a+b+v\\ &=((1/\lambda)d)+\tfrac{1}{2}(1-\lambda)(2/(1-\lambda))((1-\lambda)a+b)+\tfrac{1}{2}(1-\lambda)(2/(1-\lambda))v\end{aligned}$$

where

$$(1/\lambda)d\in\{f\}^0\cap C,$$
$$(2/(1-\lambda))((1-\lambda)a+b)\in I,$$
$$(2/(1-\lambda))v\in U.$$

Therefore w is an element of the right hand side of (*) which proves the claim.

(4.5) COROLLARY. *Let E be an OLCS such that the positive cone is normal. If I is a closed subspace with Property (I_2''), then I is an order ideal.*

Proof. It is equivalent to prove that I^0 is a perfect subspace of $(E', \sigma(E', E))$. But by (4.4), I^0 has Property (I_1''). Hence, if $f\in I^0$ then for some $g\in C'$, we have $g-2f$ is positive and $-2f\leq g-2f$. If U is any $\sigma(E',E)$ 0-neighborhood, then there exist $h\in I^0$ and $k\in U$ such that $0\leq h\leq g-2f$ and $-2f\leq h+k$. Thus we have $-(h+f)\leq -f\leq (h+f)+k$. Therefore, I^0 is perfect.

(4.6) PROPOSITION. *Let E be an* OLCS *with a normal cone that gives an open decomposition. If I is a cosed subspace with Property* (I_2''), *then I^0 has Property* (I_1).

Proof. Let f be positive and $g\in I^0$ such that $g\leq f$. We want to find a positive $h\in I^0$ with $g\leq h\leq f$. Let

$$V=\{x\in E\,|\ |g(x)|\leq\frac{1}{4},\ \ |h(x)|\leq\frac{1}{4}\}$$

and let U be a closed convex circled 0-neighborhood such that $2U\subseteq(V_1\cap C)-(V_1\cap C)$, where V_1 is a 0-neighborhood with $V_1+V_1\subseteq V$. We claim that

$$(*)\quad C\cap\Gamma((I-C)\cup(\{f\}^0\cap C)\cup U)\subseteq\Gamma((\{f\}^0\cup C)\cup I\cup 4V).$$

Once the claim is proved, we can take the polars to obtain

$$\frac{1}{4}V^0\cap I^0\cap(f-C')\subseteq\bar{\Gamma}(\{(I-C)^0\cap(f-C')\cap U^0\}\cup(-C')).$$

Now we note that the right hand side is contained in $I^0\cap C'\cap(f-C')\cap U^0-C'$, and since $g\in\frac{1}{4}V^0\cap I^0\cap(f-C')$, we have

$$g\leq h\leq f$$

for some positive $h\in I^0$. To finish the proof, we are left to prove the claim. Let x be an element of the right hand side of (*). Then x is positive and $x=\lambda_1(a-z)+\lambda_2 y+\lambda_3 u$, where $y\in\{f\}^0\cap C$, $z\in C$ and λ_1, λ_2 and λ_3 are nonnegative numbers with sum equal to 1. When $\lambda_1=1$ we have $x=a-z$, so that $0\leq z\leq a$. Hence, $z\in I$ since I is an order ideal by (4.5). Therefore $x\in I$ and so x is contained in the right hand side of (*). If $\lambda_1=0$, we have nothing to prove. Thus, assume $0<\lambda_1<1$. If θ is such that $2\theta\in V_1\cap C$ and $u\leq\theta$, then $0\leq\lambda_1(a-z)+\lambda_2 y+\lambda_3\theta$ so that $\hat{0}\leq\lambda_1\hat{z}\leq\lambda_1\hat{y}+\lambda_3\hat{\theta}$. Note that $0\leq\lambda_2 y+\lambda_3\theta$. By Property ($I_2''$), we obtain $b\in I$ and $2v\in\lambda_1 U$ such that $0\leq\lambda_1 z+b+v\leq\lambda_2 y+\lambda_3\theta$.

Let $w=\lambda_2 y+\lambda_3\theta-(\lambda_1 z+b+v)$. Then w is positive and $x=w+\lambda_1 a+b+v+\lambda_3(u-\theta)$ which can be written as

$$\alpha((1/\alpha)w)+((1-\beta)/2)((2/(1-\beta))(\lambda_1 a+b)+((1-\beta)/2)((2/(1-\beta))v)+(\lambda_3/2)(2(u-\theta)),$$

where $\alpha=\lambda_2+\lambda_3/2$ and $\beta=\lambda_2+\lambda_3$. Since $f(w)\leq f(\lambda_2 y+\lambda_3\theta)\leq\alpha$, we have $(1/\alpha)w\in\{f\}^0\cap C$. Thus

$$x\in\Gamma\{(\{f\}^0\cap C)\cup I\cup 4V\}$$

and the claim is proved.

The following theorem is a combination of (4.2) and (4.6).

(4.7) THEOREM. *Let E be an* OLCS *with a normal cone that gives an open decomposition. If I is a closed of E, then I has Property* (I_2'') *if and only if I^0 has Property* (I_1).

5. Subspaces with property (I_1'').

In this section, we prove by means similar to those in the preceeding section that if I is a closed subspace of an OLCS with a normal cone that gives an open decomposition, then I has Property (I_1'') if any only if I^0 has Property (I_2).

(5.1) PROPOSITION. *Let E be an* OLCS *and let I be a closed subspace of E such that I^0 has Property* (I_2). *Then the subspace I has Property* (I_1'') *in $\sigma(E, E')$.*

Proof. Let $x \in E$ be positive and $a \in I$ with $a \leq x$. Then $a \in (x-C) \cap I$. Note that it is sufficient to prove

$$(x-C) \cap I \subseteq \overline{(x-C) \cap I \cap C - C}.$$

First we show that

$$(*) \quad C' \cup \Gamma(I-C') \cup (\{x\}^0 \cap C') \subseteq \Gamma((\{x\}^0 \cap C') \cup I^0).$$

Note that every element of the left side of (*) is of the form $\lambda(g_1-g_2)+(1-\lambda)f$ which is positive and $f \in \{x\}^0 \cap C'$, $g_1 \in I^0$ and $g_2 \in C'$. If $\lambda=0$, we have nothing to prove. If $0<\lambda$, then we have $0 \leq \lambda g_2 \leq (1-\lambda)f$ with f and g in E'/I^0. By applying Property (I_2), we get $0 \leq \lambda g_2 + h \leq (1-\lambda)f$ for some $h \in I^0$. If k is the positive linear form $(1-\lambda)f-\lambda g_2-h$, then $k(x) \leq (1-\lambda)f(x) \leq (1-\lambda) < 1$. Let α be $k(x)$ if $k(x) \neq 0$ and $\alpha=\frac{1}{2}$ when $k(x)=0$. Then

$$\lambda(g_1-g_2)+(1-\lambda)f = \alpha((1/\alpha)k) + (1-\alpha)((1/(1-\alpha))(\lambda g_1 + h),$$
$$(1/\alpha)k \in \{x\}^0 \cap C', \quad \lambda g_1 + h \in I^0.$$

Thus $\lambda(g_1-g_2)+(1-\lambda)f \in \Gamma((\{x\}^0 \cap C') \cap I^0)$. Therefore, (*) holds. Now, by taking polars of both sides of (*), we obtain

$$\begin{aligned}(\{x\}^0 \cap C')^0 \cap I &= (x-C) \cap I \\ &\subseteq (C' \cap \Gamma\{I^0 - C'\} \cap (\{x\}^0 \cap C')\})^0 \\ &\subseteq \overline{\Gamma}(-C \cup \{I \cap C \cap (x-C)\} \\ &\subseteq \overline{I \cap C \cap (x-C) - C}.\end{aligned}$$

Thus, the proof is completed.

(5.2) COROLLARY. *Let E be an* OLCS *with a normal cone that gives an open decomposition. If I is a subspace with Property* (I_1'), *then the closure of I has Property* (I_1'') in $\sigma(E, E')$.

Proof. By (3.2), I^0 has Property (I_2). Hence, by (5.1) above, the closure of I in $\sigma(E,E')$ has Property (I_1'').

(5.3) PROPOSITION. *Let E be an* OLCS. *If I is a subspace of E with Property* (I_1''), *then I^0 has Property* (I_2'').

Proof. Let $f\in C'$ and $g\in E'$. Assume that for some h_1 and h_2 in I^0, we have $0\le g+h_1\le f+h_2$. We want to show that $g\in\overline{I^0+(f-C')\cap C'}$.

$$\text{Let } U=\{x\in E \mid |g(x)|\le\tfrac{1}{2},\ |h(x)|\le\tfrac{1}{2}\}.$$

First we show

$$(*)\quad I\cap\Gamma\{(\{f\}^0\cap C)\cup(-C)\}\subseteq\Gamma\{(I\cap C\cap\{f\}^0)\cup(\{I+U\}\cap(-C))\cup U\}.$$

If a is an element of the left hand side of (*), then $a\in I$ and $a=\lambda x-(1-\lambda)z$, where x and z are positive, $x\in\{f\}^0$ and $0\le\lambda\le1$. If $\lambda=0$ or $\lambda=1$, we have nothing to prove. Assume that $0<\lambda<1$ and apply Property (I_1'') to get $a\in I$, $b\in I$ and $u\in\frac{1}{2}(1-\lambda)U$ such that $a\le b+u$ and $0\le b\le\lambda x$. Then we have,

$$\begin{aligned}a&=b-(b+u-a)+u\\&=\lambda((1/\lambda)b)-\tfrac{1}{2}(1-\lambda)((2/(1-\lambda))(b+u-a)+u,\end{aligned}$$

where $(1-\lambda)b\in I\cap C\cap\{f\}^0$, $(2/(1-\lambda))(b+u-a)\in(I+U)\cap C$ and $(2/(1-\lambda))u\in U$. Therefore a is in the right hand side of (*). Now, take the polars of (*) to obtain,

$$U^0\cup(\{I+U\}\cap(-C))^0\cap(I\cap C\cap\{f\}^0)^0\subseteq\overline{\Gamma}(I^0\cap(f-C')\cap C')).$$

Clearly we have

$$f+I^0-C'\subseteq(I\cap C\cap\{f\}^0)^0,$$
$$\tfrac{1}{2}U^0\cap I^0+C'\subseteq((I+U)\cap(-C))^0.$$

Note that

$$g\in(f+I^0-C')\cap(\tfrac{1}{2}U^0\cap I^0+C').$$

From these, we obtain

$$g\in\overline{I^0+(f-C')\cap C'},$$

which completes the proof.

(5.4) THEOREM. *Let E be an* OLCS *and let I be a subspace of E. If I has Property* (I_1'') *then the closure of I has Property* (I_1'') *in* $(E,\sigma(E,E'))$ *and if the subspace has Property* (I_2''), *then the closure has Property* (I_2'') *for the weak topology.*

Proof. If the subspace I has Property (I_1''), then by (5.3), I^0 has Property (I_2'') and hence by (4.4), $I^{00}=\overline{I}$ has Property (I_1'') for $\sigma(E,E')$.

The similar proof for the Property (I_2'') is omitted.

(5.5) PROPOSITION. *Let E be an OLCS with a normal cone that gives an open decomposition. If a subspace I of E has Property* (I_1''), *then I^0 has Property* (I_2).

Proof. Let $f\in C'$ and $g\in E'$ such that $0\leq g+h_1\leq f+h_2$ for some h_1 and h_2 in I^0. We shall show that $g\in I^0+(f-C')\cap C'$. Let U be the 0-neighborhood

$$U=\{x\in E\,\big|\,|g(x)|\leq\frac{1}{2},\ |h_1(x)|\leq\frac{1}{2}\}$$

and let V be a 0-neighborhood such that $2V\subseteq U\cap C-U\cap C$. Then we claim that

$$(*)\quad I\cap\Gamma\{(\{f\}^0\cap C)\cup(-C)\cup V\}$$
$$\subseteq\Gamma\{(I\cap C\cap\{f\}^0)\cup(I+U\cap(-C))\cup U\}.$$

For the moment assume the claim is true. Then by taking the polars we obtain

$$U^0\cap(\{I+U\}\cap(-C))^0\cap(I\cap C\cap\{f\}^0)^0$$
$$\subseteq\bar{\Gamma}(I^0\cup\{(f-C')\cap C'\cap V^0\}\subseteq I^0+(f-C')\cap C'\cap V^0$$

Now, note that

$$g\in U^0$$
$$g\in f+I^0-C'\subseteq(I\cap C\cap\{f\}^0)^0,$$
$$g\in\frac{1}{2}U^0\cap I^0+C'\subseteq(I+U\cap(-C))^0.$$

Therefore we have $g\in I^0+(f-C')\cap C'$, which is what was to be proved.

To finish the proof, we are left to show the claim (*). Assume that a is an element of the left hand side of (*). Then $a=\lambda_1x-\lambda_2z+\lambda_3u$ for some x with $f(x)\leq 1$, $z\in C$, $u\in V$, $\lambda_i\geq 0$ for $i=1,2,3$ and $\lambda_1+\lambda_2+\lambda_3=1$. When $\lambda_1=0$ or 1, the element a is clearly in the right hand side of (*). Hence, assume $0<\lambda_1<1$ and let $\alpha=\lambda_1+\frac{1}{2}\lambda_3$. Then $0<\alpha<1$. Since $a\leq\lambda_1x+\lambda_3w$ for some $u\leq w\in\frac{1}{2}U\cap C$, if we apply Property ($I_1''$), we obtain $b\in I$ and $v\in((1-\alpha)/2)V$ such that $a\leq b+v$ and $0\leq b\leq\lambda_1x+\lambda_3w$. We now have

$$a=b-(b+v-a)u$$
$$=\alpha((1/\alpha)b)-\frac{1}{2}(1-\alpha)((2/(1-\alpha))(b-a+v))+v,$$

where $(1-\alpha)b\in I\cap C\cap f^0$, since

$$f(b)\leq f(\lambda_1x+\lambda_3w)\leq\lambda_1+\frac{1}{2}\lambda_3=\alpha,$$

and $(2/(1-\alpha))v\in V\subseteq U$. Note also that $(2/(1-\alpha))(b+a+v)\in I+U$. Therefore a is in the right hand side of (*) and the claim is proved.

(5.6) THEOREM. *Let E be an* OLCS *with a normal cone that gives an open decomposition. Then a closed subspace I of E has Property* (I_1'') *if and only if I^0 has Property* (I_2).

Proof. This is the combination of (5.1) and (5.5).

6. Relationship with other properties.

We have seen in previous sections, that in an OLCS with a normal cone that gives an open decomposition, the following implications are true for closed subspaces:

$$\text{Property } (I_2') \Longrightarrow \text{Property } (I_2'').$$
$$\text{Property } (I_1') \Longrightarrow \text{Property } (I_1'').$$

In this section, we consider further implications for other order properties of subspaces. A summary of these implications along with the duality relations is shown by a diagram at the end of this section.

(6.1) PROPOSITION. *Let E be an* OLCS *with a generating cone. If a subspace I of E has Property* (I_1''), then the subspace is perfect.

Proof. Let a be an arbitrary element of E and let U be a 0-neighborhood in E. Since E is positively generated, we have a positive $x\in E$ such that $2a\leq x$. By applying the Property (I_1'') for $-2a\leq x-2a$, we find that there exist $b\in I$ and $u\in U$ such that $0\leq b\leq x-2a$ and $-2a\leq b+u$. Thus $-(a+b)-u\leq a\leq (a+b)$. Therefore I is perfect.

(6.2) PROPOSITION. *Let E be an* OLCS *with a normal cone that gives an open decomposition. If a subspace I of E has Property* (I_2'') *then the subspace I has Property* (N).

Proof. Let U be a 0-neighborhood in E. We want to find a 0-neighborhood V in E such that $[(I+V)\cap C]\subseteq I+U$. First we find a 0-neighborhood W such that $[W]+[W]\subseteq U$ and set $V=W\cap C-W\cap C$. Then V is a 0-neighborhood by hypothesis. Now we claim $[(I+V)\cap C]\subseteq I+U$. For if $x\in[(I+V)\cap C]$, then $0\leq x\leq a+v$ for some $a\in I$ and $v\in V$. We may assume that $v\in W\cap C$. Then v is positive and $0\leq x\leq v$. By applying Property (I_2''), we conclude that there exist $b\in I$ and $u\in W$ such that $0\leq x+b+u\leq v$. Hence $x+b+u\in[W]$. Therefore $x\in I+[W]+[W]\subseteq I+U$.

(6.3) REMARK. If in addition, E is normed and the subspace I is closed, then the above proposition follows from (4.6).

(6.4) LEMMA. *Let E be an* OLCS *such that the positive cone gives an open* *decomposition. If a subspace I of E has both Property* (I_1'') *and Property* (I_2''), *then the subspace has the following Property* (A):

Property (A) *Let x and y be positive and let $\hat{0}\leq\hat{w}\leq\hat{x},\hat{y}$. Then for every 0-neighborhood U, there exist $a\in I$, $u\in U$ and $v\in U$ such that $0\leq w+a+u\leq x+v,\ y+v$.*

Proof. We may assume that U is convex. Let V be a convex 0-neighborhood such that $V\subseteq\frac{1}{4}(U\cap C-U\cap C)$. If x, y and w are as above, we apply Property (I_2'') twice to find a and b in I, and v_1 and v_2 in V such that

$$0\leq w+a+v_1\leq x,\ \ 0\leq w+b+v_2\leq y.$$

If u_1 and u_2 are in $\frac{1}{4}U\cap C$ such that $v_1\leq u_1$ and $v_2\leq u_2$, then we have

$$-a,\ -b\leq w+(u_1+u_2).$$

By Property (I_1''), there exist $d\in I$ and $v_3\in V$, such that

$$-b\leq d\leq w+(u_1+u_2),\ \ -a\leq d+v_3.$$

Let $u_3\in\frac{1}{4}U\cap C$ such that $v_3\leq u_3$. Then we have

$$-a,\ \ -b\leq d+u_3\leq w+(u_1+u_2+u_3).$$

Consequently,

$$0\leq w-d+(u_1+u_2)\leq w+a+(u_1+u_2+u_3),\ \ w+b+(u_1+u_2+u_3).$$

If $c_1=u_1+u_2$ and $c_2=u_1+u_2+u_3$, then c_1 and c_2 are in U and

$$0\leq w-d+c_1\leq x+c_2\leq y+c_2.$$

(6.5) LEMMA. *Let E and I be as in* (6.4). *If y is positive and if $x\leq y$ satisfies*

$$\hat{0},\hat{x}\leq\hat{z}\leq\hat{y},$$

then for any 0-neighborhood U in E, there exist $a\in I$, c_1 and c_2 in $U\cap C$ such that

$$-c_2,\ \ x-c_2\leq z+a-c_1\leq y.$$

Proof. Note that $y-x$ is positive and

$$\hat{0}\leq\hat{y}-\hat{z}\leq\hat{y},\ \ \hat{y}\leq\hat{x}.$$

Apply Lemma (3.4) to find $a\in I$, c_1 and c_2 in $U\cap C$ such that

$$0\leq y-z-a+c_1\leq y+c_2,\ \ y-x+c_2.$$

Then we have

$$-c_2,\ \ x-c_2\leq z+a-c_1\leq y.$$

(6.6) THEOREM. *Let E be an ordered Banach space with a generating normal cone. If a closed subspace I of E has both Properties* (I_1'') *and* (I_2''), *then it satisfies both Properties* (I_1') *and* (I_2').

Proof. By Proposition (3.5), it is equivalent to show the following: let x and y be positive and

$$\hat{0}\leq\hat{w}\leq\hat{x},\hat{y},$$

then for each 0-neighborhood V there exist $a\in I$ and $u\in V$ such that

$$0\leq w+a\leq x+u,\ y+u.$$

To prove the above, let x, y and w be as above and let U be the closed unit ball in E. If ϵ is an arbitrary positive number, then by (6.4), there exist $a_1\in I$, $c_1\in(\epsilon/2^3)U\cap C$ and $d_1\in(\epsilon/2^3)U\cap C$ such that

$$0\leq w+a_1+c_1\leq x+d_1, y+d_1.$$

Now we have

$$0,\ w+a_1\leq w+a_1+c_1, \text{ and}$$
$$0,\ \hat{w}+a\leq\hat{w}+a_1+\hat{c}_1.$$

By Lemma (6.5), there exist $a_2\in I$, $c_2\in(\epsilon/2^4)U\cap C$ and $d_2\in(\epsilon/2^4)U\cap C$ such that

$$-d_2,\ w+a_1-d_2\leq w+a_2-c_2\leq w+a_1+c_1.$$

Hence, we have

$$c_2-d_2\leq a_2-c_1\leq c_1+c_2.$$

By repeating the same process, we find that

$$0,\ w+a_2\leq w+a_2+d_2+c_2$$

and

$$0,\ \hat{w}+\hat{a}_2\leq\hat{w}\leq\hat{w}+\hat{a}_2+\hat{d}_2+\hat{c}_2.$$

By (6.5), there exist $a_3\in I$, $c_3\in(\epsilon/2^5)U\cap C$ and $d_3\in(\epsilon/2^5)U\cap C$ such that

$$-d_3,\ w+a_2-d_3\leq w+a_3-c_3\leq w+a_2+d_2+c_3.$$

Hence we have

$$c_3-d_3\leq a_3-a_2\leq c_2+c_3+d_2.$$

Now, by induction, we obtain

$$0,\ w+a_n\leq w+a_n+d_n+c_n$$

and there exist $a_{n+1}\in I$, $c_{n+1}\in(\epsilon/2^{n+3})U\cap C$ and $d_{n+1}\in(\epsilon/2^{n+1})U\cap C$ such that

$$-d_{n+1},\ w+a_n-d_{n+1}\leq w+a_{n+1}-c_{n+1}\leq w+a_n+d_n+c_n.$$

Thus, we have

$$c_{n+1}-d_{n+1}\leq a_{n+1}-a_n\leq d_n+c_n+c_{n+1}.$$

The generated sequence a_n is a Cauchy sequence in I and the limit a must be in I since I is closed. We also obtain, from $-d_{n+1}\leq w+a_{n+1}-c_{n+1}$, that $0\leq w+a$, since $c_n\to 0$ and $d_n\to 0$.

It is clear that we now have

$$w+a_{n+1}-c_{n+1},\ w+a_n+d_n+c_n\leq w+a_1+c_1+\sum_{i=2}^{n}d_i+2\sum_{i=2}^{n}c_i.$$

Hence, if $d_0=\sum_{i=2}^{*} d_i$ and $c_0=\sum_{i=2}^{*} c_i$, then we have

$$w+a_{n+1}-c_{n+1}\leq w+a_1+c_1+d_0+2c_0$$

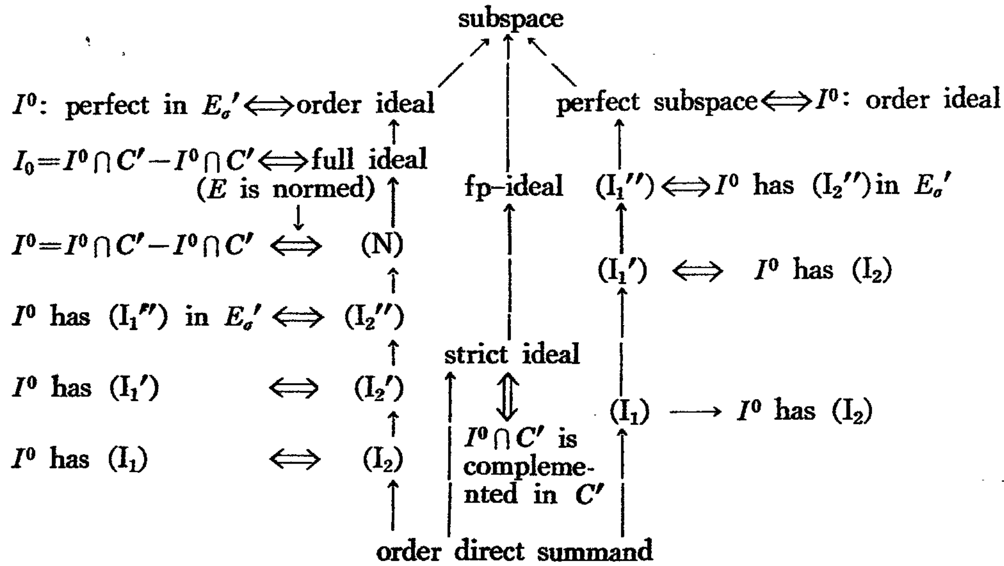
for all n. Therefore, we obtain

$$w+a\leq w+a_1+c_1+d_0+2c_0\leq x+d_1+d_0+2c_0,\ y+d_1+d_0+2c_0.$$

If we take $u=d_1+d_0+2c_0$, then $u\in\epsilon U$ and

$$0\leq w+a\leq x+u,\ y+u.$$

The following schematic diagram illustrates the inter-relations among the various order properties for subspaces that we have studied.



In the preceeding diagram, E is an ordered locally convex space with a normal cone that gives an open decomposition and I is a closed subspace of E. It should be noted that some of the above implications are true in any ordered locally convex space.

References

1. Asimow, L., *Complementary cones in dual Banach spaces*, preprint (U.C.L.A.).
2. Combes, F. & Perdrizet, F., *Certains idéaux dans les espaces vectoriels ordonnés*, Trans. Amer. Math. Soc. **131** (1968), 544-555.
3. Ellis, A.J., *Perfect order ideals*, J. London Math. Soc. **40** (1965), 288-294

4. Jameson, G. J. O., *The duality of pairs of wedges*, Proc. London Math. Soc. (3) **24** (1972), 531-547.
5. Nagel, R. J., *Ideals in ordered locally convex spaces*, Math. Scand. **29** (1971), 259-271.
6. Peressini, A. L., *Ordered topological vector spaces*, Harper and row, New York (1967).
7. Stormer, Erling, *On partially ordered vector spaces and their duals, with applications to simplexes and C*-algebras*, Proc. London Math. Soc. (3) **18** (1968), 245-265.

Korea Atomic Energy Research Institute