韓國數學敎育學會誌〈數學敎育〉
1979, 6. Vol. XVII No. 2

Journal of the Korea Society of Mathematical Education
Jun. 1979. Vol. XVII No. 2

Conditioned Galton-Watson Process on the Event $\{n+k\leqq T\}$

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In this paper, we study the conditioned limit laws on the events $\{n+k\leq T<\omega\}$ in Galton-Watson process.

The notations and terminologies used in this paper follow those of [1].

We have conditioned the Galton-Watson process Z_n on the event $\{n<T<\omega\}$, where T is the extinction time. It is meaningful to condition on $\{n+k<T<\infty\}$, $k\geq 0$; namely, the event that the process not extinct at time $n+k$ but does eventually die out, and to condition on $\{n+k=T\}$, $k\geq 0$; namely, the event that the process is extinct at time $n+k$ and $\{Z_{n+k-1}=0\}$ for all n, k.

As in [1] and [3], we have the following. When $p_1=0$ and $q=0$,

$$\lim_{n\to\infty} P(Z_n=j\mid n+k<T<\infty)=\frac{\pi_j[q^j-f_k^j(0)]}{\gamma^k\mathcal{P}_{(q)}}=b_j(k)\geq 0 \tag{1}$$

and (with $m=1$) we have

$$\lim P(Z_n=j\mid n+k=T)=\frac{\pi_j[f_k^j(0)-f_{k-1}^j(0)]}{\sum_i\pi_i[f_k^i(0)-f_{k-1}^i(0)]}=\theta_j(k)\geq 0 \tag{2}$$

where $\pi_j=\lim\limits_{n\to\infty}\dfrac{P_n(1,j)}{P_n(1,1)}$, $j\geq 1$.

We have the following proposition.

Proposition: If $p_1>0$, and $q>0$, we have

(i) If $m=1$, then $b_j(k)=0$ (3)

(ii) If $m\neq 1$, then $b_j(k)=1$ (4)

(iii) If $m=1$, then $\theta_j(k)=1$ (5)

Remark; This proposition is given in [1] without a detailed proof. We sketch the proof of this proposition: (i) If $m=1$, then $q=1$ and $f'(q)=\gamma=m=1$, therefore from (1), we see that

$$\gamma^k\mathcal{P}(q)=\gamma^k\mathcal{P}(1)=\sum\pi_j=\infty, \text{ and } \pi_j[q^j-f_k^j(0)]<\infty,$$

so we get $b_j(k)=0$. Summing for j (1) and (2), we can get (ii) and (ii).

Theorem I. If $p_1>0$, and $q>0$, we have

(i) If $m\neq 1$, $b_j(k)=\dfrac{b_j[1-f_k^j(0)]}{1-\mathcal{B}[f_k(0)]}$ (6)

(ii) If $m=1$, $\theta_j(k)=\dfrac{b_j[f_k^j(0)-f_{k-1}^j(0)]}{\mathcal{B}[f_k(0)]-\mathcal{B}[f_{k-1}(0)]}$ (7)

where $b_j=\lim\limits_{n\to\infty}P(Z_n=j\mid n<T<\infty)$.

Proof. (i) The case $m<1$, we can get in [1] easily. Now we consider the case $m>1$. Let

$$\mathcal{B}^{(b)}_{n,k}(s)\equiv E(s^{Z_n}|n+k<T<\infty)$$

$$=\sum_{j=1}^{\infty} s^j P(Z_n=j|n+k<T<\infty)$$

$$=\frac{\sum_{j=1}^{\infty} P(Z_n=j)[f_k^j(q)-f_k^j(0)]s^j}{\sum_{j=1}^{\infty} P(Z_{n+k}=j)q^j} \quad \text{(see [3] p. 1474)}$$

$$=\frac{\sum_{j=1}^{\infty} P(Z_n=j)q^j s^j-\sum_{j=1}^{\infty} P(Z_n=j)f_k^j(0)s^j}{f_{n+k}(q)-f_{n+k}(0)}$$

$$=\frac{f_n(sq)-f_n[sf_k(0)]}{q-f_{n+k}(0)}$$

$$=\frac{f_n^*(s)-f_n^*[sf_k(0)]}{1-f_{n+k}^*(0)}, \quad \text{where } f_n^*(s)=\frac{f_n(sq)}{q} \tag{8}$$

Therefore by similar method as the case $m<1$, we get

$$\mathcal{B}^{(b)}(s)\equiv\lim_{n\to\infty} \mathcal{B}^{(b)}_{n,k}(s)=\frac{\mathcal{B}(s)-\mathcal{B}[(sf_k(0)]}{1-\mathcal{B}[f_k(0)]} \tag{9}$$

(cf. [1] p. 16-p. 17).

From (9), comparing the coefficients of power expansion, we can get (6).

(ii) $\mathcal{B}^{(e)}_{n,k}(s)\equiv E(s^{Z_n}|n+k=T)$

$$=\sum_{j=1} s^j P(Z_n=j|n+k=T)$$

$$=\sum_{j=1} S^j\frac{P_n(1,j)[f_k^j(0)-f_{k-1}^j(0)]}{\sum_{i=1}^{\infty} P_n(1,i)[f_k^i(0)-f_{k-1}^i(0)]}$$

$$=\frac{f_n[sf_k(0)]-f_n[sf_k(0)]}{f_n[f_k(0)]-f_n[f_{k-1}(0)]} \tag{10}$$

From (10), we get

$$\mathcal{B}^{(e)}(s)\equiv\lim_{n\to\infty} \mathcal{B}^{(e)}_{n,k}(s)=\frac{\mathcal{B}[sf_k(0)]-\mathcal{B}[sf_{k-1}(0)]}{\mathcal{B}(f_k(0)]-\mathcal{B}[f_{k-1}(0)]} \tag{11}$$

Comparing the coefficients of (11), we get (7).

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Remark: From (2), we get the probabilistic interpretation of π_j's as

$$\pi_j=\frac{\theta_j(k)}{f_k^j(0)-f_{k-1}^j(0)}\sum_i \pi_i[f_k^i(0)-f_{k-1}^i(0)]$$

$$=\sum_i \pi_i\left[\frac{f_k^i(0)-f_{k-1}^i(0)}{f_k^j(0)-f_{k-1}^j(0)}\right]\theta_j(k) \tag{12}$$

In [1], we can see the relation between the sequence $\{\pi_j\}$, $\{\nu_j\}$ and $\{b_j\}$ and their generating functions $\mathcal{P}(s)$, $Q(s)$ and $\mathcal{B}(s)$ as following.

If $m\neq1$ and $p_1>0$, then

$$\mathcal{B}(s)=\frac{\mathcal{P}(qs)}{\mathcal{P}(q)} \tag{13}$$

$$Q(s)=Q(0)+\frac{\mathcal{P}(s)}{\mathcal{P}'(q)} \tag{14}$$

Moreover if $m<1$, then

$$\mathscr{B}(s)=1-\frac{Q(s)}{Q(0)} \tag{15}$$

From these relations, we get the following results.

Theorem Ⅱ. If $m\neq 1$, $p_1>0$, and $q>0$, then

(i) $$\frac{\nu_j}{\pi_j}=\frac{Q(q)-Q[qf_k(0)]}{\mathscr{P}(q)-\mathscr{P}[qf_k(0)]} \tag{16}$$

Moreover, $m<1$, then

(ii) $$\frac{\nu_j}{\pi_j}=\frac{Q[f_k(0)]}{\mathscr{P}'(1)\nu_0+\mathscr{P}[f_k(0)]} \tag{17}$$

(iii) $$\frac{\nu_j}{\pi_j}=\frac{Q[f_k(0)]}{\mathscr{P}[f_k(0)]-\mathscr{P}(1)} \tag{18}$$

(iv) $$\sum_{j=1}^{\infty}\pi_j=-\mathscr{P}'(1)\nu_0 \tag{19}$$

Proof. (i) From (9) and (13), we get

$$\mathscr{B}^{(b)}(s)=\frac{\mathscr{P}(sq)-\mathscr{P}(sqf_k(0)]}{\mathscr{P}(q)-\mathscr{P}[qf_k(0)]} \tag{20}$$

and we get

$$b_j(k)=\frac{\pi_j q^j[1-f_k^j(0)]}{\mathscr{P}(q)-\mathscr{P}[qf_k(0)]} \tag{21}$$

From (14) and (20), we get

$$\mathscr{B}^{(b)}(S)=\frac{Q(sq)-Q[sqf_k(0)]}{Q(q)-Q[qf_k(0)]} \tag{22}$$

and we get

$$b_j(k)=\frac{\nu_j q^j[1-f_k^j(0)]}{Q(q)-Q[qf_k(0)]} \tag{23}$$

From (21), (23), we get (16).

(ii) If $m<1$, from (9) and (15), we get

$$\mathscr{B}^{(b)}(s)=\frac{Q[sf_k(0)]-Q[s]}{Q[f_k(0)]} \tag{24}$$

and we get

$$b_j(k)=\frac{\nu_j[f_k^j(0)-1]}{Q[f_k(0)]} \tag{25}$$

From (24) and (14), we get

$$\mathscr{B}^{(b)}(s)=\frac{\mathscr{P}[sf_k(0)]-\mathscr{P}(s)}{\mathscr{P}'(1)\nu_0+\mathscr{P}[f_k(0)]} \tag{26}$$

and we get

$$b_j(k)=\frac{\pi_j[f_k^j(0)-1]}{\mathscr{P}'(1)\nu_0+\mathscr{P}[f_k(0)]} \tag{27}$$

From (25) and (27), we get (17).

(iii) From (21) and (25), we get (18).

(iv) From (17) and (18), we get the equation (19).

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References

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