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Application of a Mean Value Theorem for Integrals to prove the formula $\Sigma\frac{1}{K^2}=\frac{\pi^2}{6}$.

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I. Introduction

Expanding $f(x)=x^2$ $[x\in(0,2\pi)]$ in a Fourier series if the period is 2π, then, we get easily

$$f(x)=x^2=\frac{4\pi^2}{3}+\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx-\frac{4\pi}{n}\sin nx\right).$$

At $x=0$, above series reduces to $\frac{4\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}$.

By the Dirichet conditions the series converges to

$$\frac{1}{2}(0+4\pi^2)=2\pi^2$$

at $x=0$.

Then
$$\frac{4\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}=2\pi^2$$

and so

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Another proofs of the formula

$$\Sigma\frac{1}{K^2}=\frac{\pi^2}{6} \tag{1}$$

is given by several types [3], [4].

This note wants to give for clear cut proof of formula (1) by means of the Third Mean Value Theorem for Integrals.

Indeed Mean Value Theorem is good for proof of (1) which would avoid, e.g. convergence theorem for Fourier series.

II. Prrof of the fomula (1)

The Third Mean Value Theorem statede as follows [1], [2].

Theorem. Suppose that $g(x)$ is increasing on $[a,b]$ and that $f(x)$ is integrable on $[a,b]$, then there is a number ξ in $[a,b]$ such that

$$\int_a^b f(t)g(t)dt=g(a)\int_a^{\xi} f(t)dt+g(b)\int_{\xi}^b f(t)dt \tag{2}$$

For the Dirichlet Kernel, the representations;

$$\left.\begin{aligned} &D_n(x)=\frac{1}{2}+\sum_{k=1}^{n}\cos kx \\ \text{or} \\ &D_n(x)=\frac{\sin(2n+1)\frac{x}{2}}{2\sin\frac{x}{2}}(x\in R,\ n\in N) \end{aligned}\right\} \tag{3}$$

are known. Now we set

$$M_n=\int_0^{\pi} tD_n(t)dt$$

Using the first pepresentation of (3)

$$M_{2m-1}=\int_0^{\pi} tD_{2m-1}(t)dt$$

$$=\int_0^{\pi}\left\{t\left(\frac{1}{2}+\sum_{K=1}^{2m-1}\cos kt\right)\right\}dt$$

By partial integration of the last term can be written

$$M_{2m-1}=2\left\{\frac{\pi^2}{8}-\sum_{K=1}^{m}\frac{1}{(2K-1)^2}\right\}\ (m\in N) \tag{4}$$

On the other hand the second representation of (3) leads to

$$M_{2m-1}=\int_0^{\pi}\frac{\frac{t}{2}}{\sin\frac{t}{2}}\sin(4m-1)\frac{t}{2}dt \tag{5}$$

In (5), we apply the Third Mean Value Theorem (2) as following manner. Let the function in (2)

$$f(x)=\sin\frac{(4m-1)x}{2},$$

$$g(x)=\frac{\frac{x}{2}}{\sin\frac{x}{2}}$$

with $g(0)=1,\quad g(\pi)=\frac{\pi}{2}$.

Then statement (4) becomes

$$M_{2m-1}=2\left\{1+\left(\frac{\pi}{2}-1\right)\cos(4m-1)\frac{\xi}{2}\right\}\frac{1}{4m-1}\quad(0\leqq\xi\leqq\pi)$$

$$=O\left(\frac{1}{m}\right)\quad(m\to\infty) \tag{6}$$

A combination of (4) and (6) yields

$$\sum_{K=1}^{\infty}\frac{1}{(2K-1)^2}=\frac{\pi^2}{8} \tag{7}$$

Letting the value of sum (1) s and s′ of (7) immediately we get

$$s=\frac{1}{2^2}s+s' \quad \frac{3}{4}s=s'=\frac{\pi^2}{8}$$

$$s=\sum\frac{1}{K^2}=\frac{\pi^2}{6}$$

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