天文学会誌
The Journ. of Korean. Soc. 12, (7-9)

GENERATION OF GRAVITATIONAL WAVE IN THE SOURCE THEORY

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Received 1979 November 1

ABSTRACT

The rate of gravitational quadrupole radiation is derived in the formalism of source theory. It is also shown that gravitational superradiance is theoretically possible.

I. INTRODUCTION

We study in this paper the gravitational radiation in the phenomenological approach of source theory (Schwinger). This theory was shown to be very effective in dealing with electromagnetic radiation (Schwinger *et al.*, 1976), gravitational interaction energy (Cho *et al.*, 1976) and other processes (Kim, 1979). These results are usually obtained by applying the formula for transition rate given as the imaginary part of phase function of the so-called vacuum persistence amplitude.

For gravitational interaction the vacuum persistance amplitude has the phase integral

$$W(T) = 4\pi G \int (dx)(dx') \left[T_{\mu\nu}(x)\, D_+(x-x')\, T^{\mu\nu}(x') - \frac{1}{2} T(x)\, D_+(x-x')\, T(x') \right] \tag{1}$$

where $D_+(x-x')$ is the appropriate gravitational propagator function and the integration is four-dimensional.

$T_{\mu\nu}(x)$ is the total energy-momentum tensor and $T(x)$ is the contraction $g_{\mu\nu} T^{\mu\nu}$.

II. GRAVITATIONAL SYNCHROTRON RADIATION

We will consider the synchrotron gravitational radiation. In this case the imaginary part of the phase integral turns out

$$\mathrm{Im} W = \frac{4\pi G}{2(2\pi)^2 C^5} \int d^3x\, d^3x'\, dt\, dt' dw \cdot \frac{\sin \frac{w}{c} |\vec{x}-\vec{x}'|}{(\vec{x}-\vec{x}')} \cos W|t-t'| \cdot \left[2T^{\mu\nu}(x,t) \cdot T_{\mu\nu}(\vec{x}' \cdot t') - T(\vec{x},t)\, T(\vec{x}',t') \right] \tag{2}$$

where $T^{\mu\nu}(\vec{x},t) = m\gamma^{-1}\delta^3(\vec{x}-\vec{z}(t))\, u^\mu u^\nu$ .... (3)

$$\text{and } u^\mu = \frac{dz^\mu}{d\tau} = \gamma \frac{dz^\mu}{dt} \quad \cdots\cdots\cdots\cdots\cdots\cdots \tag{4}$$

The terms in the bracket of expression (1) can be written

$$T^{\mu\nu}T_{\mu\nu} = m^2 \gamma\gamma' c^4 \delta^3(\vec{x}-\vec{z}(t))\, \delta(x'-z(t'))\, \left(1-\frac{\vec{v}}{c}\cdot\frac{\vec{v}'}{c}\right)^2 \;\ldots\; \tag{5}$$

and

$$T\,T = m^2 c^4 \gamma\gamma' \delta^3(\vec{x}-\vec{z}(t))\, \delta^3(\vec{x}'-\vec{z}(t')) \;\ldots \tag{6}$$

And the spectrum of radiated power P (w, t) for gravitational wave is given by

$$P(w,t) = \frac{4\pi w m^2 G}{(2\pi)^2 c} \int_{-\infty}^{\infty} dt' \frac{\sin \frac{w}{c} |\vec{z}(t)-\vec{z}(t')|}{|\vec{z}(t)-\vec{z}(t')|} \cdot \cos w|t-t'| \cdot \gamma\gamma' \left[\left(1-\frac{\vec{v}}{c}\cdot\frac{\vec{v}'}{c}\right)^2 - \frac{1}{2} \right] \;\ldots \tag{7}$$

For a particle rotating in a circular orbit with frequency W_0 and at radius R, we set

$$\vec{v}(t)\cdot\vec{v}(t') = R^2 w_0^2 \cos w_0\tau \quad \cdots\cdots\cdots \tag{8}$$

and

$$|\vec{z}(t)-\vec{z}(t')| = 2K \sin \frac{w_0\tau}{2} \quad \cdots\cdots\cdots \tag{9}$$

with

$t' - t = \tau$

Putting exp. (8) and (9) into exp. (7), and carrying out the integration as Tsai and Ebber (1976) we obtain finally

$$P(w,t) = \frac{G}{c}\frac{w_0^2}{v} m^2\gamma^2 \left[\sum_{n=1}^{\infty} n \left\{ \left(\frac{1}{2} - 2\beta^2 + \beta^4\right) \right.\right.$$

$$\cdot \int_0^{\eta} J_{2n}(x)\,dx - 2\beta^2(1-2\beta^2)J'_{2n}(\eta) + 4\beta^4 J'''_{2n}(\eta)\Big\}\Big] \quad \ldots\ldots (10)$$

This is the results similar to that contained in the reference (Price *et al.* 1973). From (10) we can make approximation for various limiting cases such as high or low energy particles to obtain required formulas for appropriate power spectrum.

III. GRAVITATIONAL SUPERRADIANCE

We now consider gravitational superradiance which may be more conveniently handled in a momentum representation of the vacuum persistence amplitude. In momentum space the phase integral W (T) is given by

$$W(T) = 4\pi G \int \frac{(dk)}{(2\pi)^4} \int (dx)\, T^{\mu\nu *}(x)\, e^{ikx} \frac{1}{2} (g_{\nu\lambda} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\lambda} - g_{\mu\nu} g_{\lambda\sigma}) \cdot \frac{1}{k^2 - i\epsilon} \int (dx')\, T^{\lambda\sigma}(x') e^{-ikx'}$$

$$= \frac{4\pi G}{(2\pi)^4} \int dw\, d^3k \,\frac{1}{k_0^2 - |x|^2 - i\epsilon} \Big\{ T^{*\mu\nu}(x_{\vec{k},w}) \cdot T_{\mu\nu}(x_{\vec{k},w}) - \frac{1}{2} T^*(\vec{k},w)\, T(\vec{k},w) \Big\} \quad \ldots\ldots\ (11)$$

Hence its imaginary part Im W (T) is

$$\mathrm{Im}\, W(T) = \frac{G}{(2\pi)^2} \int \frac{d^3k}{w} \Big\{ T^{\mu\nu *}(\vec{k},w)\, T_{\mu\nu}(\vec{k},w) - \frac{1}{2} T^*(\vec{k},w)\, T(\vec{k},w) \Big\} \quad \ldots\ldots\ (12)$$

Hence the radiated power spectrum P (w, t) is obtained from the relation

$$\int dw dt \frac{P(w,t)}{w} = \frac{G}{(2\pi)^2} \int w d w d\Omega \Big\{ T^{*\mu\nu}(\vec{k},w) \cdot T_{\mu\nu}(\vec{k},w) - \frac{1}{2} T^*(\vec{k},w)\, T(\vec{k},w) \Big\} \quad \ldots\ldots\ (13)$$

now we apply this relation to the case of gravitational sources distributed uniformly in a cylinder such that

$$T^{\mu\nu}(x) = \sum_{i=1}^{n} T_i^{\mu\nu}(\vec{R}_i + \vec{r}_i t) \quad \ldots\ldots\ (14)$$

where $\vec{R}_i$ is the position of the center of the individual sources. The fourier transform $T^{\mu\nu}(\vec{k},w)$ is

$$T^{\mu\nu}(\vec{k},w) = \int (dx)\, T^{\mu\nu}(x)\, e^{-ikx}$$
$$= \sum_{i=1}^{n} \int (dx)\, T^{\mu\nu}(\vec{R}_i + \vec{r}, t)\, e^{-ikx}$$
$$= \sum_{i=1}^{n} \int d^3x\, dt\, T_i^{\mu\nu}(\vec{R}_i + \vec{r}, t)\, e^{-i(\vec{k}\cdot\vec{x} - wt)}$$
$$= \sum_{i=1}^{n} \int dr e^{+iwt} \int d^3r\, T_i^{\mu\nu}(\vec{r}, t)\, e^{-i\vec{k}\cdot r}\, e^{-i\vec{k}\cdot\vec{R}_i} \quad \ldots\ldots (15)$$

We assume n sources $T_i^{\mu\nu}$ (r, t) to be identical and equal to $T_0^{\mu\nu}$ except the phase difference of $\vec{k}_0 \cdot \vec{R}_i$. Then we can write down the total source function as

$$T^{\mu\nu}(\vec{k},w) = T_0^{\mu\nu}(\vec{k},w) \sum_{i=1}^{n} (e^{+i(\vec{k}_0 - \vec{k})\cdot\vec{R}_i}) \quad \ldots\ (16)$$

Substituting this expression for the source function into eq. (13) we obtain the radiated energy

$$E = \int p(w,t)\, dt = 2\frac{G}{(2\pi)^2} \int w^2\, d\Omega_k \Lambda_{ij,\ell m}(\hat{k}) \cdot T^{ij*}(\vec{k},w)\, T^{\ell m}(\vec{k},w) = 2\frac{G}{(2\pi)^2} \int w^2 d\Omega_k \Lambda_{ij\,\ell m}(\hat{k}) \cdot T_0^{ij*}(\vec{k},w) \Big[\sum_{i=1}^{n} (e^{i(\vec{k}_i - \vec{k})\cdot\vec{R}_i})\Big]^2 \quad \ldots\ldots (17)$$

where

$$\Lambda_{ij,\ell m}(\hat{k}) = \delta_{i\ell}\delta_{jm} - 2\hat{k}_j\hat{k}_m\delta_{ij} + \frac{1}{2}\hat{k}_i\hat{k}_j\hat{k}_\ell\hat{k}_m - \frac{1}{2}\delta_{ij}\delta_{\ell m} + \frac{1}{2}\delta_{ij}\hat{k}_\ell\hat{k}_m + \frac{1}{2}\delta_{\ell m}\hat{k}_i\hat{k}_j \ \ldots\ (18)$$

Within the quadrupole approximation we can introduce moment tensor

$$D^{ij}(t) \equiv \int d^3x\, x^i x^j T^{\mu\nu}(\vec{x}, w) \ldots\ldots\ (19)$$

And the angular distribution of radiated energy can be expressed

$$\frac{dE}{d\Omega_k} = 2\frac{Gn^2}{(2\pi)^3}\int_0^\infty dw\, w^6 \Lambda_{ij,\ell m}(\vec{k})\, D^{ij*}(w)$$

$$\cdot D^{\ell m}(w)\Gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

where Γ is defined (Rehler *et al.*, 1971)

$$\Gamma \equiv \left|\frac{1}{n}\left\{\sum_{i=1}^{n} \exp\left[i(\vec{k}_0 - \vec{k})\cdot R_i\right]\right\}_{\mu\nu}\right|^2$$

$$= \frac{1}{n} + \frac{4(n-1)}{n}\left(\frac{\sin\frac{1}{2}H(1-\cos\theta)}{\frac{1}{2}H(1-\cos\theta)}\right)^2$$

$$\cdot\left(\frac{J_1(A\sin\theta)}{A\sin\theta}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

Here H/K is the height of cylinder and A/K= γ its radius. Now the function Γ has a sharp peaking in the forward direction at $\theta = 0°$ along the axis of cylinder. The numerical calculation shows the ratio of the energy densitry at $\theta = 0°$ and $\theta = 45°$ to be almost 10^6. In addition the radiated energy is proportional to the square of the number of the sources. These are the features of superradiance in the electromagnetic radiation and our results show that superradiance may take place also in gravitational radiation.

IV. CONCLUSION

The source theory provides a convenient and adequate method of calculation for various gravitational radiative processes such as synchrotron radiation and superradiance and others within the limit of linear approximation. The theoretical results show that the gravitational superradiance is one possible way of producing intense directional gravitation wave. However its experimental feasibility and detailed numerical analysis awaits further investigation.

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