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Single-period Stochastic Inventory Problems with Quadratic Costs

Song, Moon Ho*

ABSTRACT

Single-period inventory problems such as the newspaper boy problem having quadratic cost functions for both shortages and overage are examined to determine the optimal order level under various principles of choice such as minimum expected cost, aspiration level, and minimax regret. Procedures for finding the optimum order levels are developed for both continuous and discrete demand patterns.

I. INTRODUCTION

In any decision of interest, there are two or more alternative courses of action among which the the decision maker must choose. A principle of choice then indicates which alternative is actually to be selected. [2] In inventory problems we are concerned with making optimal decisions with respect to an inventory system and in particular, with making optimal decisions that minimize the total cost of an inventory system. [3]

The purpose of this paper is to identify and relate certain principles of choice to single-period inventory problems with quadratic cost functions for both shortages and overages, in such a manner that the optimal order level could be determined under each principle of choice. Such principles of choice as expectation, aspiration level, and minimax regret will be examined in this paper.

II. THE CLASSIC SINGLE-PERIOD INVENTORY PROBLEM

This chapter is concerned with single-period inventory problems where the demand for a period is a random variable having either a known or an unknown probability distribution, and costs are linear. This problem has been studied

* 国防大学院

extensively, [1] [2] and forms the basis for the work presented in later chapters.

In an inventory situation in which items are ordered at the beginning of the period, let C_o be the unit cost of shortage (or profit per item as an opportunity cost) and let C_s be the unit cost of surplus at the end of the period. The decision variable is S, the quantity on hand at the beginning of the period. Let demand D be a random variable which denotes the demand during the period, with probability density function f (D) and distribution function F (D).

The cost per period is

$$C(S) = \begin{cases} C_s (S-D), & O \le D \le S, \\ C_o (D-S), & S < D, \end{cases}$$

or alternatively, the profit per period is

$$\pi(S) = \begin{cases} C_o D - C_s (S-D), & O \le D \le S, \\ C_o S, & S < D. \end{cases}$$

Well known results regarding the optimal order level S_0 may be summarized as follows.

A. OPTIMAL SOLUTIONS FOR CASES WHERE F(D) IS ESTIMABLE

To minimize expected cost or maximize expected profit, the optimal order level S_0 is chosen so that

$$F(S_0 - 1) < C_o / (C_s + C_o) < F(S_0),$$

if the demand is discrete, or

$$F(S_0) = C_o / (C_s + C_o),$$

if the demand is continuous. [1]

To maximize the probability that cost does not exceed the decision maker's aspiration level A, where the demand is continuous, the optimum order level S_0 is chosen so that [2]

$$f(S_0 - A/C_s) = f(S_0 + A/C_o).$$

B. OPTIMAL SOLUTIONS FOR CASES WHERE Dmax IS ESTIMABLE BUT F(D) IS NOT

To minimax regret, the optimal order level S_0 is chosen such that

$$S_0 < C_o (D_{max} + 1) / (C_s + C_o) < S_0 + 1,$$

if the demand is discrete, or

$$S_0 = C_o (D_{max}) / (C_s + C_o),$$

if the demand is continuous. [2]

The next chapter will be developed in a similar way as we seek solutions for the quadratic case.

III. SINGLE-PERIOD INVENTORY MODELS HAVING QUADRATIC COST FUNCTIONS

In order to translate a realistic inventory problem into a mathematical problem of minimizing a cost function, both flexible and simple approximations to a wide range of cost relationships are desirable to allow easy mathematical solutions. Consideration of the kinds of costs involved suggests that a U-shaped cost curve is required. [5] For example, the cost of inventory is high when inventory is large, and high also at the other extreme when inventory is so small that there are frequent runouts of inventory. Somewhere between these extremes, the combined costs are at a minimum. With these considerations in view, the cost functions may sometimes be approximated with reasonable accuracy by a positive definite quadratic form. [5]

This chapter is concerned with single-period inventory models having quadratic cost functions for both shortages and overages in connection with both continuous and discrete demand patterns.

A. CONTINUOUS DEMAND PATTERNS

The procedure developed below is a method for finding the optimal order level S_0 when the cost functions can be approximated by

$$\text{cost per period, } C(S) = \begin{cases} C_s (S-D)^2, & 0 \le D \le S, \\ C_0 (D-S)^2, & S < D. \end{cases} \qquad \cdots\cdots (1)$$

1. Minimum Expected Cost Solution

Suppose that the demand for a period is a random variable having a known probability distribution. It is assumed that the demand is continuous. The expected total cost per period of the system is

$$E(C(S)) = \int_0^S C_s (S-D)^2 f(D)\, dD + \int_S^\infty C_0 (D-S)^2 f(D)\, dD. \quad \cdots\cdots (2)$$

To find the optimal order level S_0, the expected cost function is differentiated with respect to S and the results are set equal to zero. This involves differentiation of an integral. It can be shown that if

$$F(t) = \int_{A(t)}^{B(t)} G(t, x)\, dx,$$

then

$$dF(t)/dt = \int_{A(t)}^{B(t)} \partial G(t,x)dx/\partial t + G[t,B(t)]dB(t)/dt - G[t,A(t)]dA(t)/dt.$$

This is known as Leibniz Rule. Applying this result to equation (2) yields

$$dE(C(S))/dS = 2C_S \int_0^S (S-D)\, f(D)\, dD - 2C_O \int_S^\infty (D-S)\, f(D)\, dD. \quad \cdots\cdots\cdots (3)$$

Setting the derivative equal to zero and solving leads to

$$(\bar{D} - S_O)/[\bar{D}(S_O) - S_O\ F(S_O)] = (C_O - C_S)/C_O, \quad \cdots\cdots\cdots (4)$$

where $\bar{D}$ = Expectation of demand. E (D), and

$$\bar{D}(S_O) = \int_0^{S_O} Df(D)\, dD.$$

To check whether S_O satisfying equation (4) gives a minimum, the second derivative must be examined. From equation (3),

$$d^2E(C(S))/dS^2 = 2C_S \int_0^S f(D)\, dD + 2C_O \int_S^\infty f(D)\, dD.$$

Since f (D) is a probability density, the value of the unit cost C_S and C_O. Thus the value of S_O satisfying equation (4) does indeed furnish an optimal solution.

Equation (4) is not tractable in general because of the difficulties imposed by $\bar{D}(S_O)$. For the uniform demand distribution, however, it is possible to obtain S_O and the minimum cost C^* in explicit form. This is illustrated as follows. Let the demand density be

$$f(D) = \begin{cases} 1/D_{max}, & O \le D \le D_{max}, \\ O, & \text{otherwise}. \end{cases}$$

Then $\bar{D} = D_{max}/2$ and

$$F(S_O) = \int_0^{S_O} f(D)\, dD = S_O/D_{max}$$

and

$$\bar{D}(S_O) = \int_0^{S_O} Df(D)\, dD = So^2/2D_{max}$$

Substituting these values in equation (4) gives the optimal order quantity

$$S_O = D_{max}/(1+\sqrt{C_S/C_O}). \quad \cdots\cdots\cdots (5)$$

With the uniform distribution and S_O, equation (2) gives

$$C^* = (C_S S_O{}^3 + C_O(D_{max} - S_O)^3)/3D_{max}$$

as the minimum expected cost.

2. Aspiration Level Solutions

It is possibly true that some form of aspiration level principle is the most widely used of all principles in management decision making as alternatives become increasingly expensive to discover. An aspiration level is simply some level of cost which the decision maker desires not to exceed. For the inventory problem we are considering, an aspiration level policy might be expressed as follows. For a given aspiration level, A, select the optimal order level S_0 which maximizes the probability that the cost will be equal to or less than A. [2]

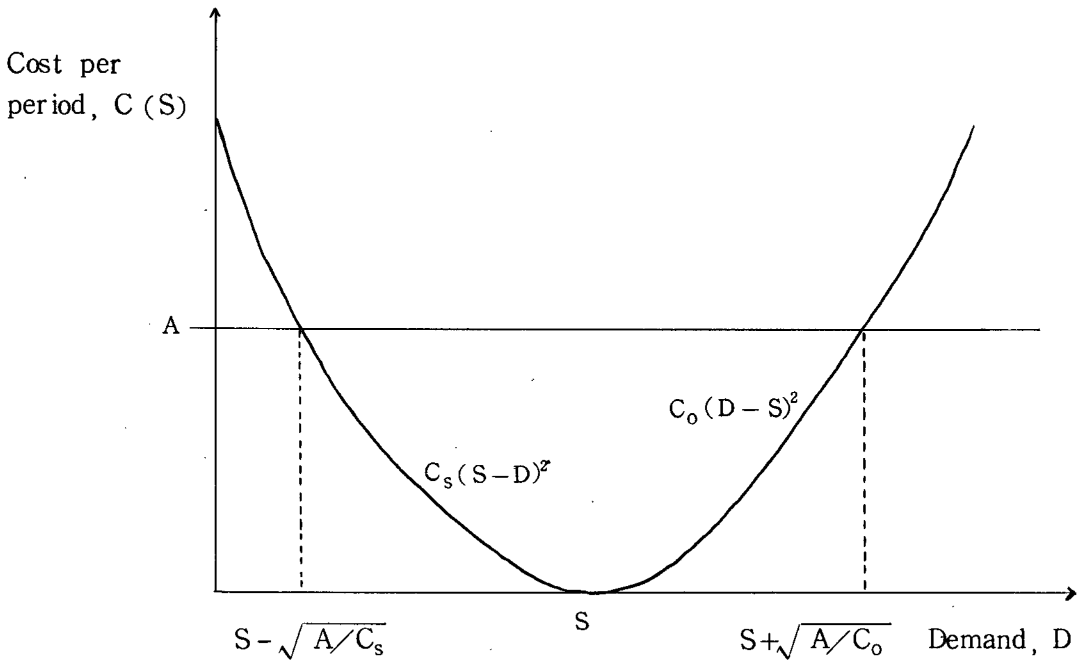

Figure 1. Quadratic Cost Function v.s. Demand, Showing Aspiration Level

From Figure 1, it follows that

$$pr\ (cost \leq A) = pr\ (S-\sqrt{A/C_s} \leq D \leq S+\sqrt{A/C_0})$$
$$= F\ (S+\sqrt{A/C_0}) - F\ (S-\sqrt{A/C_s}).$$

It is assumed that the probability density function of the demand is unimodal with respect to the maximum of that function defined on the range of demand. To find the optimal level S_0, pr (cost $\leq$ A) is differentiated with respect to S and the results are set equal to zero :

$$dpr\ (cost \leq A)/DS = f\ (S+\sqrt{A/C_0}) - f\ (S-\sqrt{A/C_s}) = o.$$

This leads to

$$f\ (S_0 + \sqrt{A/C_o}) = f\ (S_0 - \sqrt{A/C_s})$$

as our basis for choosing an optimal value of S.

It can be easily shown that if f (D) is unimodal and symmetric about $\bar{D}$ (as, for example, the normal distribution), then

$$S_0 = \bar{D} + \frac{1}{2}(\sqrt{A/C_s} - \sqrt{A/C_o}).$$

This is shown in Figure 2.

It f (D) is unimodal with mode at o (as, for example, the exponential distribution) , then

$$S_0 = \sqrt{A/C_s}$$

as the aspiration level solution. This is shown in Figure 3. We note that in the latter case it is unnecessary to estimate the unit outage cost C_o to achieve an optimal solution.

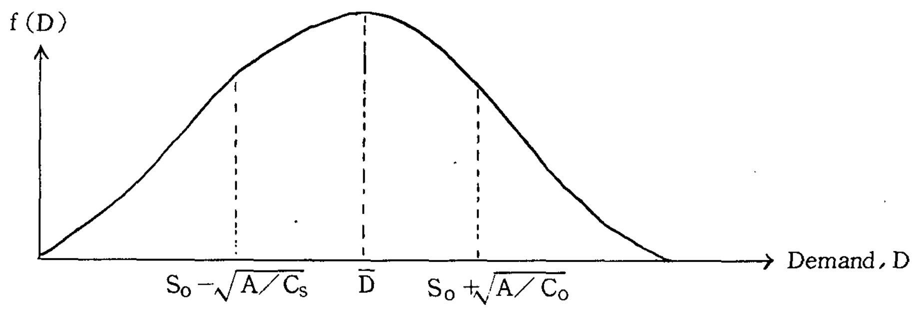

Figure 2. Application of the Aspiration Level Decision Rule to a Unimodal, Symmetric Probability Distribution for Demand .

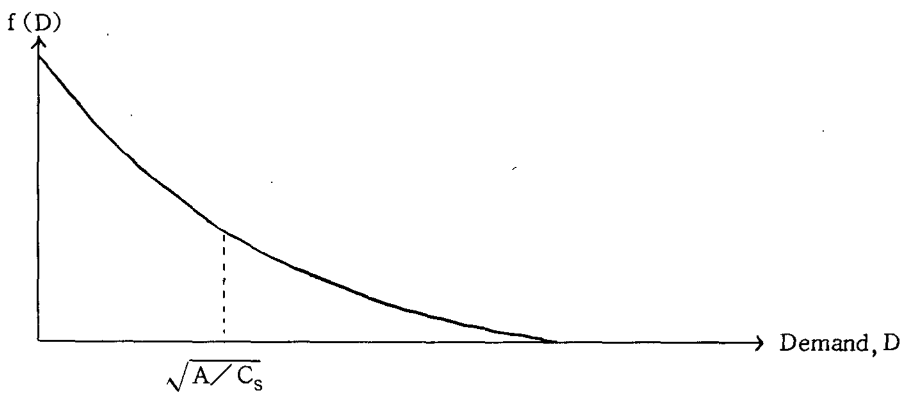

Figure 3. Application of the Aspiration Level Decision Rule to a Unimodal Probability Distribution for Demand Having Mode at O.

3. Minimax Regret Solutions

A decision for which the analyst elects to consider several possible futures, whose probabilities cannot be estimated is called a decision under uncertainty. [2] When we are unable to estimate f (D), the inventory problem may be treated as a decision under uncertainty. A principle of choice for decisions under uncertainty has been proposed by L. J. Savage, who suggests that a new matrix called a " regret matrix " be computed first. For each possible future combination of demand and order level, the difference should be computed between the actual cost that will occur and the best cost that could be occurred for the future under consideration. This difference is called "regret." [4]

This principle of choice may be appropriate for cases where the maximum demand, D_{max}, is known but f (D) is not. By observing the cost function and bearing in mind that demand takes on any value between o and D_{max}, it should be easily seen that the minimum cost will be zero for any given order level S, when demand turns out to be the order level. The maximum surplus cost will be $C_s S^2$, when demand is zero. The maximum shortage cost will be $C_o (D_{max} - S)^2$, when demand turns out to be D_{max}. For any order level S the maximum cost will simply be the maximum of these two quantities and so will the maximum regret. It will be noted that the maximum surplus cost $C_s S^2$ will increase as order level increases and the maximum shortage cost will decrease as order level increase. The minimax cost will occur when they are equal, thus S_o satisfying the equation

$$C_s S_o^2 = C_o (D_{max} - S_o)^2$$

will be the optimal order level. Solving this for S_o we have

$$S_o = D_{max} / (1 + \sqrt{C_s / C_o})$$

as the minimax regret solution. It should be noted that this result agrees with equation (5) which gives the minimum expected cost solution when demand is uniformly distributed between o and D_{max}. Accordingly, for the case where f (D) cannot be estimated, the same result would occur if we had used Laplace's Principle of Insufficient Reason which suggests computing a simple average when probabilities cannot be estimated. [2]

B. DISCRETE DEMAND PATTERNS

Suppose that the demand is a discrete random variable having a known probability distribution. When demand D and order level S are constrained to discrete units o, u, 2u, ⋯ and p (D) is the probability mass function of demand, then the expected total cost of the system becomes

$$E(C(S)) = C_s\sum_{D=0}^{S}(S-D)^2 p(D) + C_o\sum_{D=S+u}^{\infty}(D-S)^2 p(D) \quad \cdots\cdots (6)$$

The necessary conditions for S_o to be the optimal order level are

$$E(C(S_o+u)) - E(C(S_o)) \geq o \quad \cdots\cdots (7)$$

$$E(C(S_o-u)) - E(C(S_o)) \geq o \quad \cdots\cdots (8)$$

To find the conditions for the system, the differences

$$E(C(S+u)) - E(C(S))$$

and

$$E(C(S-u)) - E(C(S))$$

must be evaluated in general first. Here,

$$E(C(S+u)) - E(C(S)) = 2uC_s\sum_{D=0}^{S}(S-D+\frac{1}{2}u)p(D)$$

$$-2uC_o\sum_{D=S+u}^{\infty}(D-S-\frac{1}{2}u)p(D)$$

and

$$E(C(S-u)) - E(C(S)) = -2uC_s\sum_{D=0}^{S}(S-D-\frac{1}{2}u)p(D)$$

$$+2uC_o\sum_{D=S+u}^{\infty}(D-S+\frac{1}{2}u)p(D).$$

Applying these resulte in equations (7) and (8) and solving leads to

$$L(S_o) \leq (C_o - C_s)/C_o \leq U(S_o), \quad \cdots\cdots (9)$$

where

$$L(S_o) = (S_o - \frac{1}{2}u - \bar{D})/(F(S_o)(S_o - \frac{1}{2}u) - \bar{D}(S_o)),$$

$$U(S_o) = (S_o + \frac{1}{2}u - \bar{D})/(F(S_o)(S_o + \frac{1}{2}u) - \bar{D}(S_o)),$$

and

$$\bar{D}(S_o) = \sum_{D=0}^{S_o} Dp(D).$$

It can be easily shown that these are also sufficient conditions for S_o to be the optimal order level.

As an example, suppose $u=5$, $C_s = \$3$, $C_o = \$30$, and the demand for the item for a period is random with probabilities $p(o) = 0.05$, $p(5) = 0.25$, $p(10) = 0.35$, $p(15) = 0.30$, and $p(20) = 0.05$. Computation of the values of $L(S)$ and $U(S)$ are summarized in Table I. Since

$$(C_o - C_s)/C_o = 0.9,$$

the optimal order quantity is

$$S_o = 15,$$

because

$$0.857 = L(15) < 0.9 = (C_o - C_s)/C_o < U(15) = 0.98.$$

As a check on this result, $E(C(S))$ of equation (6) is calculated for all values of S. This is done in the last column of Table I. As expected, the least cost of the system occurs when $S = 15$.

No rule is given for an aspiration level solution in the discrete demand case because it is straightforward to compute pr (cost ≤ A) for each order quantity S, and choose the one which maximizes pr (cost ≤ A). To this, we will consider the previous example. Let A, the aspiration level for cost, be $1,000. Then the optimal order level S_0 is easily found as 15 units from Table II.

TABLE I

Tabulation of L (S) and U (S)

D, S	P(D)	F (S)	D (S)	L (S)	U (S)	E C(S)
0	0.05	0.05	0	102	− 62	S 3,862.50
5	0.25	0.30	1.25	15.5	− 2.75	1,503.75
10	0.35	0.65	4.75	− 22	0.668	408.75
15	0.30	0.95	9.25	0.857	0.98	172.50
20	0.05	1.00	10.25	1.00	1.00	356.25

TABLE II

Computation of pr (cost ≦ S 1,000)

S	D p (D)	0 0.05	5 0.25	10 0.35	15 0.30	20 0.05	pr (cost ≦ $ 1,000)
0		0	750	3,000	6,750	12,000	0.3
5		75	0	750	3,000	6,750	0.65
10		300	75	0	750	3,000	0.95
15		675	300	75	0	750	1.00
20		1,200	675	300	75	0	0.95

In a discrete demand pattern, the Savage principle; which suggests computing a regret matrix first, may also be appropriate for the cases where D_{max} is estimable but p (D) is not. For each possible future combination of demand D and order quantity S, the difference is computed between the actual cost that will occur and the minimum cost that could be occurred. Having completed the regret matrix, the optimal order level is selected which minimizes the maximum regret.

To illustrate this, we will again consider the previous example. A regret matrix is completed first as shown in Table III, and the optimal order level S_0 is selected as 15 units, since the regret of that quantity is the minimum.

TABLE Ⅲ
Regret Matrix

S, D	0	5	10	15	20	max regret
0	0	750	3,000	6,750	12,000	12,000
5	75	0	750	3,000	6,750	6,750
10	300	75	0	750	3,000	3,000
15	675	300	75	0	750	750
20	1,200	675	300	75	0	1,200

So far, we have examined the cases where both shortage and overage cost functions are quadratic.

Ⅳ. CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER WORK

We have reviewed the classic single-period inventory problem in the second chapter and in a similar way, procedures for finding the optimal order level under various principles of choice have been developed when the cost functions are quadratic for both shortages and surplus in Chapter Ⅲ.

When demand is a continuous random variable, Leibniz Rule becomes a very useful tool in differentiating of an integral of which the integrand is quadratic and it is noted that non-negative demand distribution combined with non-negative shortage and surplus costs assures the second derivative of the expected cost function always positive and thus the minimum expected cost is easily found by simply setting the first derivative equal to zero and solving for the optimal order level S_0. When demand is a discrete random variable, the finite difference inequations become very useful tools to find the minimum expected cost solution. In a discrete demand case, no rule is given for an aspiration level solution, because it is straightforward to compute the probability that cost is less than or equal to a given aspiration level A, for each order quantity S, and choose the one which maximizes pr (cost $\leq$ A). In a continuous demand case, the assumption of the demand density being a unimodal is a crucial thing to achieve an aspiration level solution. Also, we have noted that the general rules for the minimum expected cost solution are not tractable in many cases because of the difficulties imposed by the integrals $\bar{D}(S_0)$ or $\bar{D}^c(S_0)$. We have seen, however, that for the uniform demand distribution, it is possible to find the optimal solution, which agrees with that of the minimax regret, when the maximum demand is estimable. The use of more than one principle of choice has resulted in several possible optimal solutions.

In a variety of actual contexts the cost function might be better approximated by a piece-wise linear or exponential function. It must also be pointed out that, in decisions under risk, not only the expectation, but perhaps also the variance and other parameters of the distribution should be taken into account. For example, if two different order levels have the same expected cost, then the one with the smaller variance of the expected cost will be chosen. It is recommended that further work along these lines be conducted.

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