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LOCALLY ORDER-CONVEX SPACES.

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0. Abstract

The first part of this note is concerned with a neighbourhood base characterisation of locally order-convex spaces. The notions of order*-inductive limits and order ultrabornologicity in the class of locally order-convex spaces are introduced and studied in the latter part. These are the non-convex generalisation of o-inductive limits and o-bornological spaces.

1. Introduction

A locally order-convex space is a partially ordered vector space together with a linear topology for which there exists a base of neighbourhoods of the origin consisting of order-convex and balanced subsets. Kist [3] studied locally o-convex (order-convex and convex) spaces. Iyahen [2] developed concepts of $*$-inductive limits and ultrabornological spaces in the general topological vector spaces setting. The objects of this note are to define and give some results on order $*$-inductive limits and on order ultrabornological properties of locally order-convex spaces, using analogous techniques of Iyahen [2] (thus generalising the results of Kist [3]).

In section 2, we define an analogue of a suprabarrel in a topological vector space and use it to prove some basic results on locally order-convex spaces. Section 3 is devoted to the results on the finest locally order-convex topologies making certain positive linear mappings continuous (that is, order $*$-inductive limits) and on the finest locally order-convex topologies on a partially ordered vector spaces. In the last section, we study those locally order-convex spaces E which have the property that every positive linear mapping of E with range in any locally order-convex space is continuous.

Regarding the theory of topological vector spaces we refer to Horvath [1] and for those undefined order-theoretic terms we refer to Schaefer [5].

2. Locally order-convex spaces

DEFINITION 2.1. A subset U of a partially ordered vector space (E, C) is called an *order-convex suprabarrel* (hereafter abbreviated to o-suprabarrel) if U is balanced, absorbent and order-convex and, if there exists a sequence (U_n) of balanced, absorbent and order-convex subsets of E such that $U_1+U_1\subset U$ and $U_{n+1}+U_{n+1}\subset U_n$ for all n. If, in addition, U is closed we call it an *order-convex ultrabarrel* (*o-ulrabarrel*). We call $(U_n)(n=1,2\cdots)$ a defining sequence for U.

It is immediate that balanced, absorbent, o-convex subsets of a partially ordered vector space (E,C) are o-suprabarrels. However, an o-suprabarrel need not be o-convex and need not have a defining sequence of o-convex sets. Intersection of a finite number of o-suprabarrels is an o-suprabarrel. If U is a suprabarrel in (E,C), then $[U]$, the order-convex hull of U is an o-suprabarrel. The inverse image of an o-suprabarrel by a positive linear mapping is an o-suprabarrel. The image of an o-suprabarrel by a positive linear mapping is an o-suprabarrel provided the mapping is onto.

DEFINITION 2.2. An F-semi-norm ν on a partially ordered vector space (E,C) is called *monotone* if $\nu(x)\leq\nu(y)$ whenever $0\leq x\leq y$ in E.

Let U be an o-suprabarrel with a defining sequence (U_n) $(n=1, 2\ldots)$ in a partially ordered vector space (E,C). By the method of construction on page 3 of Wealbroeck [6], we can associate an F-semi-norm ν with U, as follows; $\nu(y)=\inf\{\beta : y\in W_\beta\}$, $(y\in E)$ where $W_\beta=E$ for $\beta\geq 1$ and $W_\beta=\sum_{t_k=1} U_k$ for every dyadic rational $\beta=\sum_{k=1}^{n} t_k\, 2^{-k}$. Suppose E has the decomposition property, then W_β is order-convex, as it is the sum of order-convex sets U_k. Hence $y\in W_\beta$ and $0\leq x\leq y$ imply $x\in W_\beta$. That is, $\nu(x)\leq\nu(y)$ whenever $0\leq x\leq y$.

Thus an F-semi-norm associated with an o-suprabarrel is monotone provided E has the decomposition property.

DEFINITION 2.3. (Wong and Ng [7]) *A* linear topology τ on a partially ordered vector space (E,C) is said to be *locally order-convex* if it admits a neighbourhood base at 0 consisting of order-convex sets; in this case, we shall say (E,C,τ) is a locally order-convex space.

REMARK. Locally order-convex topologies were first considered by Namioka [4]. He called these topologies locally full.

The following proposition gives an useful characterisation of a neighbourhood base at the origin in a locally order-convex space.

PROPOSITION 2.4. *In a locally order-convex space, there exists a base of* *neighbourhoods at origin consisting of o-suprabarrels; Conversely, let (E,C) be a partially ordered vector space and let $\mathscr{U}$ be a filter base at the origin consisting of o-suprabarrels with their defining sequences. Then there exists a unique vector topology τ on E for which E is a locally order-convex space and for which $\mathscr{U}$ is a base of τ-neighbourhoods at the origin.*

Proof is straightforward.

COROLLARY 2.5. *The collection of all o-suprabarrels in a partially ordered vector space (E,C) is a neighbourhood base for the finest locally order-convex topology on E.*

For certain class of partially ordered topological vector spaces, the notion of locally order-convexity is equivalent to a condition in terms of continuons F-semi-norms. We shall make this precise in the next theorem, but first we require a lemma due to Namioka [4, p. 19].

LEMMA 2.6. *Let (E,C,τ) be a partially ordered topological vector space. Then the following are equivalent.*

1. *The space (E,C,τ) is locally order-convex;*
2. *Given a τ-neighbourhood U of zero, there exists a τ-neighbourhood V of zero such that $0 \leq x \leq y$ for some y in V implies $x \in U$.*

THEOREM 2.7. *Let (E,C,τ) be a partially ordered topological vector space with decomposition property. Then the following statements are equivalent.*

1. *τ is a locally order-convex topology.*
2. *The family of all τ-continuous monotone F-semi-norms determines the topology τ.*

PROOF. (1)$\Rightarrow$(2). Let $\{\nu_i\}$ $(i \in I)$ be the family of all τ-continuous monotone F-semi-norms, and τ' be the topology generated by $\{\nu_i\}$ $(i \in I)$. It is easy to see that τ' is coarser than τ. We now show that τ is coarser than τ'. Let U be a balanced τ-neighbourhood of the origin. Since τ is locally orderconvex, there exists an o-suprabarrel V contained in U. By the remark preceding definition 2.3, the F-semi-norm ν_V of V, is τ-continuous and monotone. Also, $\{x \in E : \nu_V(x) < 1\} \subset V \subset U$; so U is a τ'-neighbourhood, as required.

(2)$\Rightarrow$(1). Let U be a τ-neighbourhood of 0. Then there exists a finite number $\{\nu_i\}$ $(i=1, 2, \cdots, n)$ of monotone F-semi-norms such that $V=\{x \in E : \max_{i=1,2,\cdots,n} \nu_i(x) < \varepsilon ; 0 < \varepsilon < 1\} \subset U$. V satisfies the property (2) of Lemma 2.6.

and so (E, C, τ) is locally order-convex,

It is useful to note that there is a method available, for constructing locally order-covex topologies from vector topologies on partially ordered vector spaces. We shall not describe it here but refer to [7, p. 56).

We conclude this section with a remark on the finest locally order-convex topology. Let e be an order-unit in a partially ordered vector space (E, C). Then the set $[-e, e]$ is balanced, convex, and absorbing. Hence the Minkowski functional ν_e of $[-e, e]$ is a semi-norm on E. Kist in [3] observed that the topology τ_e induced by ν_e on E is the finest locally o-convex topology. We claim that τ_e coincides with the finest locally order-convex topology τ on E; in fact, if V is any balanced order-convex τ-neighbourhood of the origin in E, then V is absorbing. So there exists a $\lambda>0$ such that $\lambda e \in V$, implying $\lambda[-e, e] \subset V$ Hence V is a τ_e-neighbourhood.

3. Order $*$-inductive limits

Let (E, C) be a partially ordered vector space, and (E_i, C_i, τ_i) a family of locally order-convex spaces, $(i \in I)$. Let f_i be a positive linear mapping from E_i into E for each $i \in I$. Then the order $*$-inductive limit (hereafter abbreviated to o-$*$-inductive limit) topology on E with respect to the family $((E_i, C_i, \tau_i) ; f_i)$ is defined to be the finest locally order-convex topology on E for which all the positive linear mapping f_i's are continuous.

PROPOSITION 3.1. *τ always exists on E.*

PROOF. Let $\mathscr{L}$ be the set of all locally order-convex topologies on E. The topology $\eta = \{\phi, E\}$ is locally order-convex and is the least element of $\mathscr{L}$. Since finite intersections of o-suprabarrels is an o-suprabarrel, the supremum of an arbitrary non-empty family of locally order-convex topologies is again locally order-convex. Let $\mathscr{L}_0$ be the subset of $\mathscr{L}$ consisting of those topologies for which each positive linear mapping f_i is continuous. $\mathscr{L}_0$ is non-empty since $\eta \in \mathscr{L}_0$, and the supremum of $\mathscr{L}_0$, which also belongs to $\mathscr{L}_0$, is obviously the required topology.

The space E equipped with the o-$*$-inductive limit topology is called the *o-$*$-inductive limit.*

We observe that the o-$*$-inductive limit topology on E is a linear topology, and so weaker than the strongest linear topology on E relative to which all the f_i's are continuous, that is, the linear $*$-inductive limit topology as defined in [2, p. 286]. Thus the two topologies coincide if and only if, the linear $*$-inductive limit topology on E is locally order-convex. At present, we do not have an example to show that the two topologies are distinct.

PROPOSITION 3.2. *Let (E_i, C_i, τ_i) $(i\in I)$ be a family of locally order-convex spaces; For each $i\in I$, let f_i be a positive linear mapping of E_i into a partially ordered vector space (E, C). Let $\mathscr{U}=\{U\}$ be the collection of all o-suprabarrels of E with the property that, for each $i\in I$, $f_i^{-1}(U)$, $f_i^{-1}(U_n)$ $(n=1, 2\cdots)$ are τ_i-neighbourhoods of 0 in E_i, where $\{U_n\}$ is a defining sequence of U. Then $\mathscr{U}$ is a base of neighbourhoods of 0 in E for the o-$*$-inductive limit topology with respect to the locally order-convex spaces $\{E_i\}$ and the positive linear mappings $\{f_i\}$.*

PROOF. Clearly, $\mathscr{U}$ forms a base of neighbourhoods of 0 in E for a locally order-convex topology τ' on E, by proposition 2.4. If $\mathscr{W}$ is a base of balanced, order-convex neighbourhoods of 0 for any other locally order-convex topology τ'' on E for which all the f_i's are continuous, then each $W\in\mathscr{W}$ is absorbing, o-suprabarrel in E. It is straightforward to check that $W\in\mathscr{U}$, and so $\mathscr{W}\subset\mathscr{U}$ from which it follows that $\tau''\subset\tau'$. Thus τ' is the strongest such topology and therefore τ' is the o-$*$-inductive limit topology on E.

COROLLARY 3.3. *If (F, η) is a locally order-convex space and if g is a positive linear mapping of E into F, then g is continuous with respect to the o-$*$-inductive limit topology τ on E if and only if, $g\circ f_i$ is continuous for each $i\in I$.*

PROOF. If g is continuous, then clearly the mappings $g\circ f_i$ are all continuous. Conversely, suppose g is a positive linear mapping such that $g\circ f_i$ is continuous for each $i\in I$. Let U_0 be any balanced order-convex η-neighbourhood of the origin in F. Choose a sequence of balanced, order-convex η-neighbourhoods $\{U_n\}$ $(n=1, 2\ldots)$ such that $U_n+U_n\subset U_{n-1}$, $(n=1, 2\ldots)$. Then $g^{-1}(U_0)$ is a balanced, absorbing o-suprabarrel in E, with a defining sequence $\{g^{-1}(U_n)\}$ $(n=1, 2\ldots)$. Also $f_i^{-1}(g^{-1}(U_n))=(g\circ f_i)^{-1}(U_n)$ is a τ_i-neighbourhood of 0 in E_i for each $i\in I$ and $n=0, 1, 2\ldots$ Thus, by proposition 2.3, $g^{-1}(U_0)$ is a τ-neighbourhood of 0 in E and so g is continuous.

Let (E, C) be a partially ordered vector space. For each $a\in E$ with $a\geq 0$, let

$$E_a=\bigcup_{t\in R^+}\{x : x\in E, -ta\leq x\leq ta\}$$

$C_a = E_a \cap C$.

Then E_a is a subspace and a is an order-unit for (E_a, C_a). Let τ_a be the locally-order-convex topology on E_a induced by the semi-norm ν_a of $[-a, a]$ in E_a. Then we have the following analogue of proposition 5.2 [3].

THEOREM 3.4. *Let (E, C, τ) be a locally order-convex space. Then τ is the finest order-convex topology on E if and only if (E, C, τ) is the o-*-inductive limit of (E_a, C_a, τ_a) for $a \in E$, with respect to inclusion mappings i_a.*

PROOF. *Necessity.* Let V be any balanced, order-convex τ-neighbourhood of 0 in (E, C). Then V is an o-suprabarrel with a defining sequence (V_n) $(n=1, 2\cdots)$, say. The sets $V \cap E_a$, $V_n \cap E_a$ for each $a \in E$, $n=1, 2\cdots$, are balanced, order-convex and absorbing. Moreover $V_n \cap E_a + V_n \cap E_a \subset V_{n-1} \cap E_a (n=2, 3\cdots)$, and $V_1 \cap E_a + V_1 \cap E_a \subset V \cap E_a$ for each $a \in E$. That is $V \cap E_a$ is an o-suprabarrel in E_a for each $a \in E$. Since τ_a is the finest locally order-convex topology in E_a, $V \cap E_a$, $V_n \cap E_a$ are τ_a-neighbourhoods of origin in E_a. Hence V is an o-*-inductive limit neighbourhood, by proposition 3.2.

Sufficiency. Let V be any o-suprabarrel in (E, C), with a defining sequence (V_n). Then it is obvious that $i_a^{-1}(V) = V \cap E_a$, $(i_a^{-1}(V_n) = V_n \cap E_a (n=1, 2\cdots))$ is an o-suprabarrel in E_a for $a \in E$, and so a τ_a-neighbourhood in E_a. Also for each $n=1, 2\cdots$, $V_n \cap E_a$ is an o-suprabarrel in E_a and hence a τ_a-neighbourhood. The proposition 3.2, now implies V is a neighbourhood in the order *-inductive limit topology. Therefore, the order *-inductive limit topology coincides with the finest order-convex topology by Corollary 2.5.

We conclude this section with an useful analogue of proposition 2.2 of [2].

THEOREM 3.5. *Let (E, C, τ) be the order *-inductive limit of a family of locally order-convex spaces (E_i, C_i, τ_i) $(i \in I)$ with respect to positive linear mappings (f_i). For each $i \in I$, let V_i be a balanced, order-convex τ_i-neighbourhood of 0 in E_i, and let U be the order-convex hull of $\bigcup_{\Phi} \sum_{i \in \Phi} f_i(V_i)$ the union being taken over all finite subsets Φ of I. Then U is a τ-neighbourhood of o in E.*

If I is countable, then as V_i runs through a base of balanced, order-convex τ_i-neighbourhoods of 0 in E_i, the order-convex hull of the above sets form a base of τ-neighbourhoods of 0 in E.

PROOF. Let $U = \bigcup_{\Phi} \sum_{i \in \Phi} f_i(V_i)$ as given in From Iyahan [2[, we know that

U is a neighbourhood in the $*$–inductive limit topology η. Since the order-convex hull $[U]$ of U is a neighbourhood in the finest order-convx topology coarser than η, it follows $[U]$ is a τ-neighbourhood. Similarly, the second part of the theorem follows from the corresponding part of proposition 2.2 of [2].

Since the $*$–inductive limit of a sequence of locally convex spaces is locally convex, and the order-convex hull of a convex set is o-convex (see Kist [3]), we have the following:

COROLLARY 3.6. *The o–$*$–inductive limit of a sequence of locally o-convex spaces, is locally o-convex, and thus coincides with the o-inductive limit.* (See Kist [3]).

4. O-ultrabornological spaces

DEFINITION 4.1. A locally order-convex space E is called *order-ultrabornological* (*o-ultrabornological*) if every bounded positive linear mapping from E into any locally order-convex space is continuous.

We conjecture that the attributes of ultrabornological and o-ultrabornological are distinct when applied to the class of locally order-convex space. But we are unable to substantiate this. Also, at present, we do not know whether an o-bornological space as defined by Kist [3], is o-ultrabornological or not. However, the class of o-ultrabornological space is non-empty, as it contains metrisable locally order-convex spaces. In particular, if the topology τ of a partially ordered topological vector space $(E.C)$ is given by a single monotone F-semi-norm, then (E, C, τ) is o-ultrabornological.

The following concept is important in the study of o-ultrabornological space.

DEFINITION 4.2. A subset B of a partially ordered topological vector space (E, C, τ) is called a *bornivorous o-suprabarrel* if B is a balanced, bornivorous, order-convex subset of E and if there exists a sequence (B_n) of balanced, bornivorous, order-convex subsets of E such that $B_1+B_1\subset B$ and $B_{n+1}+B_{n+1}\subset B_n$ for $n=1,2\cdots$.

The next theorem gives the connection between o-ultrabornological spaces and bornivorous o-suprabarrel subsets.

THEOREM 4.3. *Let τ_1 be a locally-order-convex topology on a partially ordered vector space (E, C). Then*

1. *The family of all bornivorous o-suprabarrels in (E, C, τ_1) is a base of neighbourhoods of 0 for a finer locally order-convex topology τ_2 on E.*

2. *The topologies* τ_1 *and* τ_2 *have the same bounded subsets.*

3. *The space* (E, C, τ_1) *is o-ultrabornological if and only if* $\tau_1 = \tau_2$.

4. *and this is so, if and only if every bornivorous o-suprabarrel in* (E, τ_1) *is a* τ_1*-neighbourhood of origin.*

The proofs are straightforward.

With some simple modifications of the proofs of proposition 4.1 and Theorem 4.1 of Iyahen [2], we obtain the following analogues.

PROPOSITION 4.4. *A set of positive linear mappings form an o-ultrabornological space into a locally order-convex space is equicontinuous provided that it is uniformly bounded on bounded sets.*

THEOREM 4.5. *Any o-*-inductive limit of o-ultrabornological spaces is o-ultrabornological.*

By some easy calculations, we can prove the following corollaries of Theorem 4.5.

COROLLARY 4.6. *If F is a closed subspace of an o-ultrabornological space E, then E/F is o-ultrabornological.*

COROLLARY 4.7. *If f is a continuous, open, positive linear mapping of an oultrabornological space E onto a locally order-convex space F, then F is o-ultrabornological.*

COROLLARY 4.8. *Any countable o-inductive limit of locally o-convex o-ultrabornological spaces is o-ultrabornological.*

DEFINITION 4.9. A subset A of a linear space is called *semi-convex* if there is some $\lambda \geq o$ for which $A + A \subset \lambda A$.

DEFINITION 4.10. We say that a partially ordered topological vector space is *almost order-convex* if every bounded subset is contained in some bounded set which is closed, balanced, semi-convex and order-convex.

Clearly every locally o-convex space is almost order-convex and so is any partially ordered topological vector space whose topology is given by a bounded, order-convex neighbourhood of the origin.

The next theorem, an analogue of proposition 6.3 (*e*) of Kist [3], is a partial converse of theorem 4.5.

THEOREM 4.11. *Let* (E, C, τ) *be an almost order-convex o-ultrabornological* *space, and let $\mathscr{U}$ be the class of all closed, bounded, semi-convex, balanced, and order-convex subsets of E. For each $B\in\mathscr{U}$, let E_B be the linear subspace generated by B. Then*

1. *E_B is a partially ordered vector space.*

2. *There exists a p-normed, locally order-convex topology τ_B on E_B, for a suitable $0<p\leq 1$.*

3. *(E, C, τ) is the o-*-inductive limit of (E_B, τ_B) $(B\in\mathscr{U})$ with respect to the inclusion mappings (i_B).*

PROOF. 1. Take $C_B=C\cap E_B$ as the positive cone of E_B.

2. Since B is balanced and semi-convex, there exists a $\lambda\geq 2$ such that $B+B\subset\lambda B$. Put $p=\log 2/\log\lambda$ and for $x\in E_B$, define $\nu_B(x)=\inf(|\lambda|^p : x\in\lambda B)$. It is easy to check that ν_B is a p-norm on E_B. The topology τ_B given by ν_B, is the required topology.

3. Let U be a τ-neighbourhood of 0 in E. Then for each $B\in\mathscr{U}$, $\lambda B\subset U$ for some $\lambda>0$ implying $\lambda B\subset U\cap E_B$. So $i_B : (E_B, \tau_B)\longrightarrow(E, \tau)$ is continuous for each $B\in\mathscr{U}$. Moreover, let τ_0 be any locally order-convex topology on E for which each i_B is continuous. Then it is not difficult to show that τ_0 coincides with τ as (E, τ) is almost order-convex o-ultrabornological. This completes the proof.

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