Kyungpook Math. J.
Volume 18, Number 1.
June, 1978.

# CONCAVITY PROPERTIES FOR CERTAIN LINEAR COMBINATIONS OF STIRLING NUMBERS

By Jin B. Kim and Yong M. Lee.

## 1. Abstract

This paper studies some problems suggested by Stirling numbers, and defines generalized Stirling numbers $s(n, k, r)$, $S(n, k, r)$ and proves that generalized Stirling numbers and certain linear combination of generalized Stirling numbers are strong logarithmic concave functions of $k$ for fixed $n$ and $r$.

In the notation of Riordan [6, p. 33], the Stirling numbers $s(n, k)$ and $S(n, k)$, of the the first and second kind respectively are defined by the relations

$$(x)_n = \sum_{k=1}^{n} s(n, k)x^k$$

$$x^n = \sum_{k=1}^{n} S(n, k)(x)_k$$

where $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ is the factorial power function.

DEFINITION 1. Let $r$ be a positive integer. Let $(x)_n^r = x(x-r)(x-2r)\cdots(x-nr+r)$,

$$(x)_n^r = \sum_{k=1}^{n} s(n, k, r)x^k \qquad \text{and}$$

$$x^n = \sum_{k=1}^{n} S(n, k, r)(x)_k^r$$

The numbers $s(n, k, r)$ and $S(n, k, r)$ are called respectively generalized Stirling numbers of the first and second kind. Following the numbers $C(m, k)$ and $D(m, k)$ in Jordan ([4] p.184), we define $C(m, k, r)$ and $D(m, k, r)$.

DEFINITION 2. $C(m, k, r) = \sum_{j=m+1}^{2m-k+1} (-1)^{j+k}\binom{2m-k}{j} s(j, j-m, r)$

where $C(m, k, r)=0$ for $k>m-1$, $C(1, 0, r)=-r$, $C(m, m-1, r)=(-1)^m m!\, r^m$

The results communicated in this paper were obtained in the course of an investigation partially supported by the grant from the Research and Development Fund of the Department of Higher Education of the State of New Jersey awarded to the second author.

and $C(m, 0, r)=(-1)^m 1\cdot 3\cdot 5 \cdots (2m-1)r^m$, and

$$D(m, k, r)=\sum_{j=m+1}^{2m-k+1}(-1)^{k+j}\binom{2m-k}{j} S(j, j-m, r)$$

where $D(m, k, r)=0$ for $k>m-1$, $D(1, 0, r)=r$, $D(m, m-1, r)=r^m$ and $D(m, 0, r)=1\cdot 3\cdot 5 \cdots (2m-1)r^m$.

## 2. Strong logarithmic concavity functions

Leib [5] has shown that the Stirling numbers $s(n, k)$ and $S(n, k)$ are both strong logarithmic concave functions of $k$ for fixed $n$, that is, they satisfy $[s(n, k)]^2>s(n, k-1)s(n, k+1)$ and $[S(n, k)]^2>S(n, k-1)S(n, k+1)$ for $k=2, 3, \cdots, n-1$. We generalize this inequality in the following lemma.

LEMMA 1. *Let* $\{a(n, k): k=1, 2, \cdots, n\}$ *be a sequence such that* $[a(n, k)]^2>a(n, k-1)a(n, k+1)$ *for* $2\leqq k\leqq n-1$. *Then*

(i) $a(n, k-i)a(n, k+i)>a(n, k-i-1)a(n, k+i+1)$ *for* $i\geqq 0$, $2+i\leqq k\leqq n-i-1$.

(ii) $a(n, k)a(n, k+i+1)>a(n, k-1)a(n, k+i+2)$ *for* $i\geqq 0$, $2\leqq k\leqq n-i-2$.

REMARK 1. Lemma 1-(ii) is essentially the same as (2. 22. 1) in Hardy, Littlewood and Polya ([2], p.52). Note that the expressions in (i) are symmetric. The technique of the proof of Lemma 1-(ii) is the same as that of Lemma 1-(i). Therefore we will just prove (i) only.

PROOF of Lemma 1-(i). We prove this by induction on $i$. If $i=0$, it is trivial by the assumption. Let $m$ be a fixed positive integer and we suppose that the inequality $a(n, k-i)a(n, k+i)>a(n, k-i-1)a(n, k+i+1)$ has been proved for $i<m$. From $a(n, k-m+1)a(n, k+m-1)>a(n, k-m)a(n, k+m)$, it follows that $a(n, k-m+1)a(n, k+m-1)a(n, k-m)a(n, k+m)>[a(n, k-m)]^2[a(n, k+m)]^2>a(n, k-m-1)a(n, k-m+1)a(n, k+m-1)a(n, k+m+1)$, from which we have $a(n, k-m)a(n, k+m)>a(n, k-m-1)a(n, k+m+1)$. Hence we proved (i) by induction on $i$.

Observing that the inequality $a(n, k-i)a(n, k+i)>a(n, k-i-1)a(n, k+i+1)$ implies the inequality $[a(n, k)]^2>a(n, k-i)a(n, k+i) (i\geqq 1)$, we raise a question:

What are necessary and sufficient conditions on the sequence $\{a(n, k): k=1, 2, \cdots, n\}$ for the inequality $[a(n, k)]^2>a(n, k-i)a(n, k+i) (i\geqq 1)$ implies the inequality $a(n, k-i)a(n, k+i)>a(n, k-i-1)a(n, k+i+1)$ $(i\geqq 0)$?

THEOREM 1. *If* $a(n, k)\in\{s(n, k), S(n, k)\}$, *then* $a(n, k-i)a(n, k+i)>$

$a(n, k-i-1)a(n, k+i+1)$ *for* $i \geqq 0$, $2+i \leqq k \leqq n-i-1$.

The proof follows from Lemma 1 and [5].

Since concavity property of $s(n, k)$, $S(n, k)$, $C(n, k)$, $D(n, k)$ are deduced from their recurrence relation, we might like to ask the following question:

What conditions should be imposed on $f(n, k)$ and $g(n, k)$ such that, $a(n, k) = f(n, k)\, a(n-1, k-1) + g(n, k)\, a(n-1, k)$ of a sequence $\{a(n, k) \mid k=1, 2, \cdots, n\}$ of integers with the property $[a(n, k)]^2 > a(n, k-1)\, a(n, k+1)$?

THEOREM 2. *Let a sequence* $\{a(n, k) \mid k=1, 2, \cdots, n\}$ *of integers satisfy* $a(n, k) = (\hat{x}n+\hat{c})\, a(n-1, k-1) + (\hat{z}n+\hat{d})\, a(n-1, k)$ *with* $a(1, 0) \neq 0$, $a(n, k)=0$ *for* $k > n-1$ *or* $k < 0$, *and* $\hat{x}$, $\hat{c}$, $\hat{z}$, $\hat{d}$ *are integer constants.*

*Then, we have* $[a(n, k)]^2 > a(n, k+1)\, a(n, k-1)$ *for* $1 \leqq k \leqq m-2$.

PROOF. Using induction, we prove that all the zeroes of the polynomial $P_{n-1}(x) = \sum_{k=0}^{n-1} a(n, k)x^k$ are real for $n=3, 4, 5, \cdots$. Then the desired result follows from the Newton's inequality: If the polynomial $P(x) = \sum_{k=1}^{n} c_k x^k$ has only real roots, then

$$c_k^2 > c_{k+1}\, c_{k-1} \text{ for } k=2, 3, \cdots, n-1$$

[2].

Clearly, $P_1(x)$ has real zeroes. Suppose $P_{n-2}(x)$ has all real zeroes. We want to show that $P_{n-1}(x)$ has only real zeroes. But from the definition of $P_{n-1}(x)$ and $a(n, k)$, we have

$$\begin{aligned} P_{n-1}(x) &= \sum_{k=0}^{n-1} a(n, k)x^k \\ &= \sum_{k=0}^{n-1} \{(\hat{x}n+\hat{c})\, a(n-1, k-1) + (\hat{z}n+\hat{d})\, a(n-1, k)\}\, x^k \\ &= \{(\hat{x}n+\hat{c})x + (\hat{z}n+\hat{d})\}\, P_{n-2}(x). \end{aligned}$$

This proves that $P_{n-1}(x)$ has only real zeroes.

THEOREM 3. *Let a sequence* $\{a(n, k) \mid k=1, 2, 3, \cdots, n\}$ *of integers satisfy* $a(n, k) = (\hat{x}n+\hat{y}k+\hat{c})\, a(n-1, k-1) + (\hat{z}n+\hat{w}k+\hat{d})\, a(n-1, k)$ *with* $a(n, k)=0$ *for* $k > n-1$, $k < 0$. *Moreover,*

(a) $a(1, 0) \neq 0$ *and* $\dfrac{-2(z+d)}{2\hat{x}+\hat{y}+\hat{c}}$ *is a negative real number.*

(b) $\left(z-\frac{\hat{x}\hat{w}}{\hat{y}}\right)n+\hat{d}-\frac{\hat{w}(\hat{c}+\hat{y})}{\hat{y}}\neq 0$ *for* $n=3, 4, 5, \cdots$

(c) $g(n)$ *and* $f(n)$, *which are solutions of*
$\hat{y}g(n)-\hat{y}f(n)=\hat{x}n+\hat{c}+\hat{y}$ *and* $-\hat{w}f(n)=\hat{z}n+\hat{d}$, *satisfy* $g(n)>0$, $f(n)>g(n)+n-2$ *for* $n=3, 4, 5, \cdots$, *where* $\hat{y}\hat{w}>0$, $\hat{w}|\hat{z}$, $\hat{w}|\hat{d}$, $\hat{y}|\hat{x}$, *and* $\hat{y}|\hat{c}$.

*Then the numbers* $a(n, k)$ *satisfy the inequality*

$$[a(n, k)]^2>a(n, k+1)\, a(n, k-1).$$

PROOF. We generalized the method used in [1]. Using induction, we prove that all the zeroes of the polynomial $P_{n-1}(x)=\sum_{k=0}^{n-1}a(n, k)x^k$ are real, negative and distinct for $n=3, 4, 5, \cdots$. Then the desired result follows from Newton's inequality as in the case of theorem 2. From the definition of $P_{n-1}(x)$ and from the fact that $a(n, k)=(\hat{x}n+\hat{y}k+\hat{c})a(n-1, k-1)+(\hat{z}n+\hat{w}k+\hat{d})a(n-1, k)$, we have $P_{n-1}(x)=\{(\hat{x}n+\hat{c}+\hat{y})x+(\hat{z}n+\hat{d})\}P_{n-2}(x)+(\hat{y}x+\hat{w})x\frac{dP_{n-2}(x)}{dx}$. Let us consider the rational function:

$$Q_{n-1}(x)=\frac{(\hat{y}x+\hat{w})^{g(n)}}{x^{f(n)}}\, P_{n-2}(x).$$

Assumption (c) leads to:

$$\frac{d}{dx}Q_{n-1}(x)=\frac{(\hat{y}x+\hat{w})^{g(n)-1}}{x^{1+f(n)}}\, P_{n-1}(x).$$

Clearly $P_1(x)$ has negative real zero by assumption (a). By induction hypothesis, all zeroes of $P_{n-2}(x)$ are real negative and distinct. By assumption (b), $Q_{n-1}(x)$ has $(n-1)$ distinct, negative real zeroes and one zero at $-\infty$. By Rolle's theorem $\frac{d}{dx}Q_{n-1}(x)$ has $(n-1)$ distinct, negative real zeroes. This proves that $P_{n-1}(x)$ has $(n-1)$ distinct, negative real zeroes.

## ʹ. Some properties of generalized Stirling numbers

We begin with

LEMMA 2. $s(n, k, r)=s(n, k)r^{n-k}$ *and* $S(n, k, r)=S(n, k)r^{n-k}$.

PROOF. Consider $(x)_n^r=x(x-r)(x-2r)\cdots(x-nr+r)=\frac{x}{r}\left(\frac{x}{r}-1\right)\left(\frac{x}{r}-2\right)\cdots\left(\frac{x}{r}-r+1\right)r^n=\sum_{k=1}^{n}s(n, k)\left(\frac{x}{r}\right)^k r^n=\sum_{k=1}^{n}s(n, k, r)x^k$, from which we obtain $s(n, k, r)=s(n, k)r^{n-k}$. Similarly, $S(n, k, r)=S(n, k)r^{n-k}$.

REMARK 2. Recurrence relations for the generalized Stirling numbers take the following forms (see Riordan([6], p. 33) for the recurrence relations for the Stirling numbers) $s(n+1, k, r)=s(n, k-1, r)-nrs(n, k, r)$ and $S(n+1, k, r)=S(n, k-1, r)+krS(n, k, r)$.

LEMMA 3. *$C(n, k, r)=C(n, k)r^n$ and $D(n, k, r)=D(n, k)r^n$.*

We omit the proof of Lemma 3.

REMARK 3. (see Jordan [4] or [1]). We have that $C(m+1, k, r)=-(2m-k+1)(C(m, k-1, r)+rC(m, k, r))$ and $D(m+1, k, r)=(m-k+1)rD(m, k-1, r)+(2m-k+1)rD(m, k, r)$, as recurrence relations for $C(m, k, r)$ and $D(m, k, r)$.

THEOREM 4. *Let $a(n, k, r)$ be a member of $\{s(n, k, r), S(n, k, r)\}$. Let $n\geqq 3$ and $1\leqq k$. Then*

(i) $a(n, k-i, r)\, a(n, k+i, r)>a(n, k-i-1)\, a(n, k+i+1, r)$ $(i\geqq 0,\ 2+i\leqq k \leqq n-i-1)$

(ii) $a(n, k, r)\, a(n, k+i+1, r)>a(n, k-1, r)\, a(n, k+i+2, r)$ $(i\geqq 0,\ 2\leqq k\leqq n-2-i)$

PROOF. Let $a(n, k, r)=s(n, k, r)$. Then $[a(n, k, r)]^2=[s(n, k)]^2 r^{2n-2k}> s(n, k-1)\, s(n, k+1)r^{2n-2k}=a(n, k-1, r)\, a(n, k+1, r)$. Now (i) and (ii) follow from Lemma 1 and Lieb [5].

THEOREM 5. *For $m\geqq 3$ and $k=1, 2, \cdots, m-2$,*

(i) *the numbers $C(m, k, r)$ and $D(m, k, r)$ are strong logarithmic concave functions of $k$ for fixed $m$ and $r$, that is, letting $a(m, k, r)\in\{C(m, k, r), D(m, k, r)\}$, we have $[a(m, k, r)]^2>a(m, k-1, r)a(m, k+1, r)$.*

(ii) *Letting $a(m, k, r)\in\{C(m, k, r), D(m, k, r)\}$, We have $a(m, k-i, r)\, a(m, k+i, r)>a(m, k-i-1, r)\, a(m, k+i+1, r)$ $(i\geqq 0,\ 2+i\leqq k\leqq m-i-1)$, $a(m, k, r)\, a(m, k+i+1, r)>a(m, k-1, r)\, a(m, k+i+2, r)(i\geqq 0,\ 2\leqq k\leqq m-i-2)$.*

The proof of Theorem 5 follows from Lemmas 1, 2, 3 and [1], i.e., defining $P_{m-1}(x)=\sum_{k=0}^{m-1} C(m, k, r)x^k$ and $H_{m-1}(x)=\sum_{k=0}^{m-1} D(m, k, r)x^k$ as in [1], it is easy to prove that $C(m, k, r)$ and $D(m, k, r)$ are strong logarithmic concave functions of $k$ for fixed $n$ and $r$.

West Virginia Univ.
Morgantown, W. Va. 26506
U.S.A

Trenton State College
Trenton, N.J. 08625
U.S.A

## REFERENCES

[1] J.C. Ahuja, *Concavity properties for certain linear combinations of Stirling numbers*, The Journal of the Australian Mathematical Society Vol. XV-Part 2(1973,) 145—147.

[2] G.H. Hardy, J.E. Littlewood and G. Polya, *Inequalities*, (University Press, Cambridge, 1952).

[3] L.H. Harper, *Stirling Behavior is asymptotically normal*, Ann. Math. Statist. 38 (1967), 410—414.

[4] C. Jordan, *Calculus of finite differences*, (Chelsea, New York, 1950).

[5] E.H. Lieb, *Concavity properties and a generating function for Stirling Numbers*, J. Combinatorial Theory 5(1968), 203—206.

[6] Riordan, *An introduction to Combinatorial Analysis* (Wiley, New York, 1958).