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EUCLIDEAN AND GAUSSIAN SEMIRINGS

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1. Introduction

It is well known that every Euclidean domain is Gaussian. This is usually verified by first showing that every Euclidean domain is a principal ideal domain and then showing that every principal ideal domain is Gaussian. In [1], a Euclidean semiring is defined and the structure of ideals in the semiring is given. The purpose of this paper is to prove an analogue of this well known result for Euclidean semirings. This is done by first showing that a Euclidean semiring is an "almost" principal ideal semiring and then showing that an "almost" principal ideal semiring is a Gaussian semiring. The definitions and main theorems from [1] are used in this paper.

2. Euclidean Semirings and Almost Principal Ideals

A set S together with two binary operations called addition $(+)$ and multiplication $(\cdot)$ will be called a semiring provided $(S, +)$ is an abelian semigroup with a zero, $(S, \cdot)$ is a semigroup, and multiplication distributes over addition from the left and right. It is clear that any ring is a semiring, consequently, examples of semirings are quite numerous. If S is a commutative semiring with an identity e and every nonzero $x \in S$ has an expression $x=x'+e$, then S is called a *principal semiring*.

DEFINITION. A *Euclidean semiring* is a principal semiring E together with a function $\phi: E \rightarrow Z^+$ (the nonnegative integers) satisfying the following properties:

(i) $\phi(a)=0$ if and only if $a=0$

(ii) If $a+b \neq 0$, then $\phi(a+b) \geq \phi(a)$

(iii) $\phi(ab)=\phi(a)\phi(b)$

(iv) For all a, $b \neq 0 \in E$, there exists q, $r \in E$ such that $a=qb+r$ where $r=0$ or $\phi(r)<\phi(b)$.

A Euclidean semiring is just one of many special classes of semirings. The classes of semirings to be discussed in this paper are analogues of some of the special classes of rings.

A commutative semiring S is said to be a *semi-integral domain* if S contains no zero divisors. A Euclidean semiring is a semi-integral domain. For if $ab=0$ and $b\neq 0$, then $\phi(a)\phi(b)=\phi(ab)=\phi(0)=0$. But $\phi(a)$ and $\phi(b)$ are integers and $\phi(b)\neq 0$. Consequently, $\phi(a)=0$ and it follows that $a=0$. It is clear that the set of positive integers is a semi-integral domain.

An element in a semiring is called a *unit* if it has an inverse in the semiring. Clearly, if S is a nontrivial semiring, then the set of units in S is a multiplicative group. Denote this group by S_u and let $S^*=S-\{0\}$. A semiring S is said to be a *semi-division ring* if $S_u=S^*$. If S is a commutative semi-division ring, then S is said to be a semifield. Any field is a semifield. Also, the set of nonnegative rational numbers and the set of nonnegative real numbers are semifields. The final special class of semirings is the class of almost principal ideal semirings. Recall that an ideal is called a principal ideal if it is generated by a single element, say a. Denote such an ideal by (a).

DEFINITION. An ideal A in a semiring S is said to be an *almost principal ideal* if there exists a finite set F such that $A\cup F$ is a principal ideal. A semiring S is called an *almost principal ideal semiring* if every ideal in S is almost principal.

Clearly every principal ideal domain is an almost principal ideal semiring. The set of nonnegative integers, Z^+, is an almost principal ideal semiring, as is pointed out in [1]. The aim in this section is to show that every Euclidean semiring is an almost principal ideal semiring. To do this, some preliminaries concerning units are established.

LEMMA 2.1. *Let E be a Euclidean semiring. Then $u\in E$ is a unit if and only if $\phi(u)=1$.*

PROOF. Suppose u is a unit. Then there exists $u'\in E$ such that $uu'=e$. Consequently $\phi(u)\,\phi(u')=\phi(uu')=\phi(e)=1$ and since $\phi(u)$ and $\phi(u')$ are positive integers it follows that $\phi(u)=1$. Conversely, suppose that $\phi(u)=1$. By the division algorithm, there exists $q,\ r\in E$ such that $e=qu+r$ where $r=0$ or $\phi(u)>\phi(r)$. But $1=\phi(u)>\phi(r)$ assures that $\phi(r)=0$ and thus $r=0$. Consequently $e=qu$ and u is a unit.

COROLLARY 2.2. *Let E be a Euclidean semiring. If u is a unit and $x\in E$ then u divides x.*

PROOF. Lemma 2.1 assures that $\phi(u)=1$. The division algorithm gives

$x=qu+r$ where $r=0$ or $\phi(u)>\phi(r)$. Consequently, $r=0$ and $x=qu$.

THOREM 2.3. *Let E be a Euclidean semiring. E is a semi-division ring if and only if ϕ is bounded on E.*

PROOF. Suppose E is a semi-division ring. Let $x\in E$. Then it follows that either $x=0$ or x is a unit. Hence $\phi(x)=0$ or $\phi(x)=1$ and ϕ is bounded on E. Conversely, suppose that ϕ is bounded on E. Then there exists an integer n_0 such that $\phi(x)\leq n_0$ for all $x\in E$. Let $y\in E$ such that $y\neq 0$. If $\phi(y)=k>1$, then there exists an integer n such that $\phi(y^n)=[\phi(y)]^n>n_0$, a contradiction of the assumption that ϕ is bounded on E. Conseqently, $\phi(y)=1$ and it follows that y is a unit. Hence every nonzero element of E is a unit and E is a semi-division ring.

A cancellation property relative to the function ϕ defined on a Euclidean semiring is introduced now.

DEFINITION. A Euclidean semiring is said to have ϕ-cancellation if $a, b, c, d\in E$, $a+b=c+d$ and $\phi(a)=\phi(c)$, then $\phi(b)=\phi(d)$.

It is clear that the semiring Z^+ has the ϕ-cancellation property. The following lemma is very useful.

LEMMA 2.4. *Let E be a Euclidean semiring with ϕ-cancellation. Then $\phi(a)=\phi(b)$ if and only if $a=bu$ where u is a unit.*

PROOF. If $a=bu$, then $\phi(a)=\phi(bu)=\phi(b)\phi(u)=\phi(b)\cdot 1=\phi(b)$. On the other hand, suppose $\phi(a)=\phi(b)$. The division algorithm gives $a=bq+r$ where $r=0$ or $\phi(r)<\phi(b)$. But $\phi(a)=\phi(bq+r)\geq\phi(bq)=\phi(b)\phi(q)$. Consequently, $\phi(q)=1$ and q is a unit. Thus $\phi(a)=\phi(bq)$. Now $a+0=bq+r$ and ϕ-cancellation gives $\phi(r)=0$. Hence $r=0$ and $a=bq$.

With this lemma it is easy to show that in a Euclidean semiring ϕ-cancellation implies cancellation.

Just as rings can be classified according to their groups of units, so can semirings. The two classes of semirings to be considered are (1) the class with a finite group of units and (2) the class in which every nonzero element is a unit. The latter class is just the class of semi-division rings, in which the only ideals are the trivial ideals. Hence, these, (2), are principal ideal semirings and consequently almost principal ideal semirings. Now we restrict our attention to the former class of semirings.

Our aim now is to show that every Euclidean semiring is an almost principal ideal semiring. To show that every Euclidean ring is a principal ideal ring is easy. However, as previously mentioned, every Euclidean semiring is not a principal ideal semiring (e. g., Z^+). Proving this result for semirings requires a bit more effort than for rings. The reader is referred to [1] for the definition of ideals of the form T_a and dT_a.

LEMMA 2.5. *If E is a Euclidean semiring with ϕ-cancellation and E_u is finite, then for each $d, a \in E$, dT_a is an almost principal ideal.*

PROOF. Consider $(d)-dT_a=dT_e-dT_a=d\{T_e-T_a\}=d\{E-T_a\}$. We first show that $E-T_a$ is finite. If $x\in E-T_a$, then $\phi(x)<\phi(k)$. Suppose $E-T_a$ is not finite. Then $E-T_a$ must contain an infinite subset of distinct elements, say $F=\{x_0, x_1, \cdots, x_n, \cdots\}$, such that $\phi(x_i)=\phi(x_j)=m<\phi(a)$ for all nonnegative integers i and j. Let $E_u=\{u_1, u_2, \cdots, u_t\}$. Since $\phi(x_0)=\phi(x_1)$, it follows from lemma 2.4 that $x_0=u_{i_1}x_1$ where $1\leq i_1\leq t$. Also $\phi(x_0)=\phi(x_2)$ and it follows that $x_0=u_{i_2}x_2$ where $1\leq i_2\leq t$. Continuing in this manner, we obtain $x_0=u_{i_k}x_k$ for each positive integer k. Let $s>t$, then $x_0=u_{i_s}x_s$ where $1\leq i_s\leq t$. Since E_u has only t elements, it follows that $u_{i_s}=u_{i_r}$ for some i_r with $1\leq i_r\leq t$. Hence $u_{i_s}x_s=x_0=u_{i_r}x_r=u_{i_s}x_r$ and since u_{i_s} is a unit, if follows that $x_r=x_s$. But this contradicts the fact that F contains distinct elements. Consequently, $E-T_a$ is finite. Now $E-T_a$ being finite assures that $d\{E-T_a\}$ is finite. Hence $(d)-dT_a$ is finite and $dT_a\cup\{(d)-dT_a\}=(d)$ is a principal ideal. Cosequently, dT_a is an almost principal ideal.

THEOREM 2.6. *If E is a Euclidean semiring with ϕ-cancellation and E_u is finite, then E is an almost principal ideal semiring.*

PROOF. Let A be an ideal in E such that $A\neq\{0\}$. It was shown in [1] that $A=L\cup dT_a$, where dT_a is maximal in A, $L=\{t\in A\mid\phi(t)<\phi(da)\}$, and $L\cap dT_a=\{0\}$. Let $y\in L$. Then $\phi(y)<\phi(da)$. Let g be the greatest common divisor of y and da. From [1], it follows that there exists $c\in A$ such that $gT_c\subset A$. Now dT_a is maximal in A. Consequently, $gT_c\subset dT_a$, and it follows that d divides g. Now g divides y and it follows that d divides y, i.e. $y=dx$ for some $x\in E$. Hence $y\in(d)$. Since $\phi(d)\phi(x)=\phi(dx)=\phi(y)<\phi(da)=\phi(d)\phi(a)$, it follows that $\phi(x)<\phi(a)$ and $y\in(d)-dT_a$. Therefore $L\subset(d)-dT_a$ and since $(d)-dT_a$ is finite, it follows that L is finite. Lemma 2.5 assures that dT_a is an almost principal ideal. This fact together with $dT_a \subset A = L \cup dT_a \subset (d)$ guarantees that A is an almost principal ideal. [To see this, let A' denote the complement of A in E and note that $(d)-A=(d)\cap A'$ is finite and $A\cup\{(d)-A\}=A\cup\{(d)\cap A'\} =\{A\cup(d)\}\cap\{A\cup A'\}=(d)\cap E=(d)$].

Thus the Euclidean semirings that have a finite group of units as well as those in which every nonzero element is a unit are almost principal ideal semirings.

3. Almost Principal Ideals and Gaussian Semirings

Every Euclidean semiring with ϕ-cancellation is an almost principal ideal semiring, as was shown previously. Next we show that in a Euclidean semiring with ϕ-cancellation (i) the cancellation law holds, and (ii) an almost principal ideal has a finite basis. For if $ax=ay$, then $\phi(a)\phi(x)=\phi(ax)=\phi(ay)=\phi(a)\phi(y)$ and it follows that $\phi(x)=\phi(y)$. Lemma 2.4 assures that $y=xu$, u a unit. Consequently, $ax=ay=a(xu)=(ax)u$ and it follows that $u=e$. Thus $y=xu=xe=x$ and (i) is shown. Now if A is an ideal in a Euclidean semiring E, then $A=L \cup dT_a$, where $L=\{t\in A \mid \phi(t)<\phi(da)\}$. From [1], we know that A has basis B such that ϕ is bounded on B. Consequently, the proof of Lemma 2.5 can be used to show that B is finite.

A semigroup G is called *Gaussian* if (i) G is commutative, has an identity and satisfies the cancellation law, and (ii) every non-unit of G has an essentially unique factorization into irreducible elements.

DEFINITION. A semiring S is called *Gaussian* if its semigroup of nonzero elements is Gaussian.

It is an easy matter to prove that a commutative semigroup G with an identity and cancellation law is Gaussian if G contains no infinite proper ascending chain of principal ideals and every pair of elements in G has a greatest common divisor. (The proof may be found in [3]). This fact is needed to prove the next theorem.

THEOREM 3.1. *Let S be an almost principal ideal semiring with an identity. If the cancellation law holds in S, then S is a Gaussian semiring.*

PROOF. Let $(a_1)\subset(a_2)\subset(a_3)\subset\cdots\subset(a_n)\subset\cdots$ be an ascending chain of principal ideals in S and $\cup(a_i)=D$. Clearly, D is an ideal in S. For if x_1, $x_2\in D$

then $x_1 \in (a_q)$ and $x_2 \in (a_t)$. Suppose $q \leq t$. Then x_1, $x_2 \in (a_t)$ and it follows that $x_1 + x_2 \in (a_t)$ and $yx_1 \in (a_t)$ for any $y \in D$. This assures that D is an ideal. Since S is an almost principal ideal semiring, there exists a finite set F such that $D \cup F = (d)$ for some $d \in S$. Now D has a finite basis, say $\{d_1, d_2, \cdots, d_k\}$. Consequently, each $d_i \in (a_{\alpha_i})$ for some α_i. Let p be max $\{\alpha_1, \alpha_2, \cdots, \alpha_t\}$. Then $\{d_1, d_2, \cdots, d_t\} \subset (a_p)$ and it follows that $D \subset (a_p)$. But $(a_p) \subset D$ and consequently, $D = (a_p)$. Now if $m \geq p$, then $(a_m) \subset D = (a_p) \subset (a_m)$ and $(a_m) = (a_p)$. Therefore, the chain of ideals is finite. Next, suppose a, $b \in D$ and let (a, b) be the ideal generated by a and b. Then $(a, b) = \{ax + by \mid x, y \in D\}$. Again, since D is almost principal, there exists a finite set F' such that $(a, b) \cup F' = (c)$. Now (a)$\subset (c)$ and $(b) \subset (c)$ hence c divides a and c divides b. If $r \in D$ and r divides both a and b then $(a) \subset (r)$ and $(b) \subset (r)$. Consequently, $(a, b) \subset (r)$ and since F' is finite, it follows that $(c) \subset (r)$. Thus r divides c and c is the greatest common divisor of a and b. This proves that S is a Gaussian semiring.

THEOREM 3.2. *If E is a Euclidean semiring with ϕ-cancellation, then E is a Gaussian semiring.*

PROOF. Theorem 2.6 assures that E is an almost principal ideal semiring. The remarks at the beginning of this section assure that the cancellation law holds in E and that every ideal in E has a finite basis. Thus by Theorem 3.1 E is a Euclidean semiring.

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