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ON PURE-HIGH SUBGROUPS OF ABELIAN GROUPS

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In this paper we investigate what groups G can be represented in the form $\mathrm{Hext}_p(B, A)$ for suitable groups A and B. Theorem 1 shows that such groups must necessarily be direct sum of cyclic groups of the same order p. Furthermore, we discuss in this paper the class of groups all of whose pure-high extensions by torsion groups are splitting.

A sub-group H of G is a pure-high sub-group of G if it is maximal disjoint among pure subgroups. The exact sequence $0\longrightarrow A\longrightarrow G\longrightarrow B\longrightarrow 0$ is a pure-high extension if and only if there exists a subgroup K of G such that A is maximal disjoint from K and $(A+K)/K$ is pure in G/K. A high subgroup of a group is a pure-high subgroup of that group, and that if A is pure-high in X, $A^1=0$ and X/A is divisible, then A is a high subgroup of X. Furthermore, if B is divisible and $A^1=0$ then $\mathrm{Hext}_p(B, A)=\mathrm{Hext}(B, A)$. The elements of $\mathrm{Hext}_p(B, A)$, that is, the pure-high exact sequences can be described in a manner analogous to the definition of high sequences. In general we adopt the notations used in [1].

Before establishing the main theorem we prove the following lemma.

LEMMA. *For an elementary p-group A the following hold.*

(a) $Hext_p(\prod_{q\in P} Z(q),\ A)\cong Hext_p(\bigoplus_{q\in P} Z(q),\ A)$

(b) $Hom\ (Q\oplus \prod_{q\in P} Z(q),\ A)=0$

(c) $Hext_p(Q\oplus \prod_{q\in P} Z(q), A)=0$

PROOF. $\bigoplus_{q\in P} Z(q)$ coincides with the maximal torsion subgroup of $\prod_{q\in P} Z(q)$ and the factor group $\prod_{q\in P} Z(q)/\bigoplus_{q\in P} Z(q)$ is divisible (see theorem 9.2 and exercise 9.14 respectively of [2]). It is easy to verify that the sequence

$$0\longrightarrow \bigoplus_{q\in P} Z(q)\longrightarrow \prod_{q\in P} Z(q)\longrightarrow \prod_{q\in P} Z(q)/\bigoplus_{q\in P} Z(q)\longrightarrow 0$$

is pure-high exact. Theorem 6 of [3] implies that the sequence

$$\mathrm{Hext}_p(\prod_{q\in P} Z(q)/\bigoplus_{q\in P} Z(q),\ A)\longrightarrow \mathrm{Hext}_p(\prod_{q\in P} Z(q),\ A)\longrightarrow \mathrm{Hext}_p(\bigoplus_{q\in P} Z(q), A)\longrightarrow 0$$

is exact. Since every elementary p-group is a direct sum of cyclic groups of the same order p and does not contain elements of infinite height it follows that $A^1=0$. Also $\prod_{q\in P} Z(q)/\bigoplus_{q\in P} Z(q)$ is divisible and hence by [3]

$$\begin{aligned}\mathrm{Hext}_p(\prod_{q\in P} Z(q)/\bigoplus_{q\in P} Z(q),\ A)&=\mathrm{Hext}\ (\prod_{q\in P} Z(q)/\bigoplus_{q\in P} Z(q),\ A)\\ &\subset \mathrm{Pext}(\prod_{q\in P} Z(q)/\bigoplus_{q\in P} Z(q),\ A)\\ &\subset \mathrm{Next}(\prod_{q\in P} Z(q)/\bigoplus_{q\in P} Z(q),\ A)\\ &=0\end{aligned}$$

since A is an elementary p-group. This is in accordance with a result of Hauptfleisch [4] who proved that if $A=Z(n)=\{\alpha\}$ then $f\in T(A, B)$ if and only if $\sum_{i=1}^{p-1} f(\alpha,\ i\alpha)\in nB$. Consequently, if A is a cyclic group of prime order p then $F'(A,\ B)=T(A,\ B)$ so that $\mathrm{Next}(A,\ B)=0$ when A is an elementary p-group.

To prove (b), $\mathrm{Hom}(Q\oplus \prod_{q\in P} Z(q),\ A)\cong \mathrm{Hom}(Q,\ A)\oplus \mathrm{Hom}(\prod_{q\in P} Z(q),\ A)$ $\mathrm{Hom}(Q,\ A)=0$, since Q is divisible and A is reduced. Furthermore, the exactness of the sequence

$$0\longrightarrow \bigoplus_{q\in P} Z(q)\longrightarrow \prod_{q\in P} Z(q)\longrightarrow \prod_{q\in P} Z(q)/\bigoplus_{q\in P} Z(q)\longrightarrow 0$$

implies the exactness of the sequence

$$0\longrightarrow \mathrm{Hom}(\prod_{q\in P} Z(q)/\bigoplus_{q\in P} Z(q),\ A)\longrightarrow \mathrm{Hom}(\prod_{q\in P} Z(q),\ A)\longrightarrow \mathrm{Hom}(\bigoplus_{q\in P} Z(q),\ A)$$

(See theorem 44.4 of [1]). The first group is 0, since $\prod_{q\in P} Z(q)/\bigoplus_{q\in P} Z(q)$ is divisible and A is reduced. The last group is also 0, since $\bigoplus_{q\in P} Z(q)$ is a q-group and A is a p-group.

To prove (c), again

$\mathrm{Hext}_p(Q\oplus \prod_{q\in P} Z(q),\ A)\cong \mathrm{Hext}_p(Q,\ A)\oplus \mathrm{Hext}_p(\prod_{q\in P} Z(q),\ A)$. Since $A^1=0$ and Q is divisible it follows from [3] that $\mathrm{Hext}_p(Q,\ A)=\mathrm{Hext}(Q,\ A)\cup \mathrm{Pext}(Q,\ A)\subset \mathrm{Next}(Q,\ A)=0$ because A is an elementry p-group. Also by lemma, part (a), and theorem 7 of [3].

$$\begin{aligned}&\mathrm{Hext}_p(\prod_{q\in P} Z(q),\ A)\cong \mathrm{Hext}_p(\bigoplus_{q\in P} Z(q),\ A)\cong\\ &\prod_{q\in P} \mathrm{Hext}_p(Z(q),\ A)=\prod(\bigcap_q q\ \mathrm{Pext}(Z(q),\ A))\end{aligned}$$

since $Z(q)$ is torsion and is contained in

$$\prod(\bigcap_q q\ \mathrm{Next}\ (Z(q),\ A))=0.$$

Next we prove what groups G can be represented in the form $\mathrm{Hext}_p(B,\ A)$

for suitable groups A and B.

THEOREM 1. *Every elementary p-group is the $Hext_p$-group.*

PROOF. Imbed Z in the divisible group Q: $0\longrightarrow Z\overset{\alpha}{\longrightarrow}Q$. Define a monomorphism $\bar{\alpha}$ of Z into $Q\oplus\prod_{q\in P}(Z/qZ)$ by $\bar{\alpha}: z\longrightarrow(\alpha z, (\cdots, z+qz, \cdots))$, $z\in Z$. Then $\bar{\alpha}Z$ is a neat subgroup of $Q\oplus\prod_{q\in P}(Z/qZ)$, for if r is any prime number and $(\bar{z}, (\cdots, z_p+qZ, \cdots))\in Q\oplus\prod_{q\in P}(Z/qZ)$ such that $r(\bar{z}, (\cdots, z_p+qZ, \cdots))=\bar{\alpha}z=(\alpha z, (\cdots, z+qZ, \cdots))$ then $r\bar{z}=\alpha z$ and $r(\cdots, z_p+qZ, \cdots)=(\cdots, z+qZ, \cdots)$. From the second equality it follows that $z\in rZ$, say $z=rz'$, $z'\in Z$. Hence

$$\begin{aligned}\bar{\alpha}z=(\alpha z, (\cdots, z+qZ, \cdots))&=(\alpha rz', (\cdots, rz'+qZ, \cdots))\\ &=r(\alpha z', (\cdots, z'+qZ, \cdots))\\ &=r(\bar{\alpha}z')\in r(\bar{\alpha}Z)\end{aligned}$$

proving that $\bar{\alpha}Z$ is a neat subgroup of $Q\oplus\prod_{q\in P}Z/qZ$ which is isomorphic to $Q\oplus\prod_{q\in P}Z(q)$, and $\bar{\alpha}Z$ is maximal disjoint from $\bigoplus_{q\in P}Z(q)$. This is in accordance with the remark of Harrison [3] section 4, that is A is a neat subgroup of G if and only if A is maximal disjoint from some subgroup K of G.

It is easy to see that $\bar{\alpha}Z$ is pure in $Q\oplus\prod_{q\in P}Z(q)$. Furthermore, $\bigoplus_{q\in P}Z(q)$ is the maximal torsion subgroup of $\prod_{q\in P}Z(q)$ and therefore of $Q\oplus\prod_{q\in P}Z(q)$. Hence $\bar{\alpha}Z+\bigoplus_{q\in P}Z(q)$ is pure in $Q\oplus\prod_{q\in P}Z(q)$ (see exercise 5, page 116 of [1]). By lemma 26.1 of [1] we have $(\bar{\alpha}Z+\bigoplus_{q\in P}Z(q))/\bigoplus_{q\in P}Z(q)$ is pure in $(Q\oplus\prod_{q\in P}Z(q))/\bigoplus_{q\in P}Z(q)$. From theorem 4 of [3] it follows that the sequence

$$0\longrightarrow Z\bar{x}\overset{\bar{\alpha}}{\longrightarrow}Q\oplus\prod_{q\in P}Z(q)\longrightarrow(Q\oplus\prod_{q\in P}Z(q))/\bar{\alpha}Z\longrightarrow 0$$

is pure-high exact. Theorem 6 of [3] implies that for an elementary p-group G the sequence

$$\begin{aligned}\mathrm{Hom}(Q\oplus\prod_{q\in P}Z(q), G)\longrightarrow\mathrm{Hom}(Z, G)\longrightarrow\mathrm{Hext}_p((Q\oplus\prod_{q\in P}Z(q))/\bar{\alpha}Z, G)\\ \longrightarrow\mathrm{Hext}_p(Q\oplus\prod_{q\in P}Z(q), G)\end{aligned}$$

is exact. The first and the last groups are 0 by lemma, and since $\mathrm{Hom}(Z, G)\cong G$ it follows

$$G\cong\mathrm{Hext}_p((Q\oplus\prod_{q\in P}Z(q))/\bar{\alpha}Z, G).$$

This proves that a group which is direct sum of cyclic groups of the same order p is a Hext_p-group.

We shall now study the class of groups all of whose pure-high extensions by torsion groups are splitting.

A group G is called *a* H_p^t*-group* if $\mathrm{Hext}_p(T, G)=0$ for all torsion groups T.

THEOREM 2. *A necessary and sufficient condition for a group G to be a* H_p^t*-group is that* $\mathrm{Hext}_p(Z(p^\infty), G)=0$ *for all prime numbers p.*

PROOF. Only sufficiency needs verification. Since a torsion group is the direct sum of p-groups, it is sufficient to prove the result for p-group. If T is any p-group then we have the existence of a pure-high exact sequence $0 \longrightarrow H \longrightarrow T \longrightarrow T/H \longrightarrow 0$ with H, direct sum of cyclic groups and $T/H = \oplus Z(p^\infty)$. The sequence $\mathrm{Hext}_p(T/H, G) \longrightarrow \mathrm{Hext}_p(T, G) \longrightarrow \mathrm{Hext}_p(H, G)$ is exact. H is torsion; hence $\mathrm{Hext}_p(H, G) = \bigcap_p p\,\mathrm{Pext}(H, G)=0$ for H is the direct sum of cyclic groups. Also

$$\mathrm{Hext}_p(T/H, G)=\mathrm{Hext}_p(\bigoplus_{p\in P} Z(p^\infty), G)= \prod_{p\in P} \mathrm{Hext}_p(Z(p^\infty), G)=0.$$

It is easy to prove that a direct product $\prod G_i$ is a H_p^t-group if and only if each G_i is.

The following theorem gives more insight into H_p^t-groups.

THEOREM 3. *Let* $0 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 0$ *be pure-high exact then following hold.*

(a) *If both H and G/H are* H_p^t*-groups, then so is G.*

(b) *If G is a* H_p^t*-group, then so is H whenever the factor group is reduced.*

(c) *G is a* H_p^t*-group if and only if H is a* H_p^t*-group whenever G/H is torsion free.*

PROOF. (a), (b) and (c) follow from the exact sequences $\mathrm{Hext}_p(Z(p^\infty), H) \longrightarrow \mathrm{Hext}_p(Z(p^\infty), G) \longrightarrow \mathrm{Hext}_p(Z(p^\infty), G/H)$, $(Z(p^\infty), G/H) \longrightarrow \mathrm{Hext}_p(Z(p^\infty), H) \longrightarrow \mathrm{Hext}_p(Z(p^\infty), G)$ and $\mathrm{Hom}(Z(p^\infty), G/H) \longrightarrow \mathrm{Hext}_p(Z(p^\infty), H) \longrightarrow \mathrm{Hext}_p(Z(p^\infty), G) \longrightarrow \mathrm{Hext}_p(Z(p^\infty), G/H)$ respectively.

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