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AUTOMORPHISMS OF QUASI-ASSOCIATIVE ALGEBRAS

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This note is to show that the group of automorphisms of a quasi-associative algebra of characteristic $\neq 2$ consists of the automorphisms of an associative algebra with involution $*$ which commute with $*$.

Let D be an associative algebra over a field Φ of characteristic $\neq 2$ and $\lambda \in \Phi$. Define a new multiplication $\cdot$ on the vector space of D by $x \cdot y = \lambda xy + (1-\lambda)yx$ in terms of the associative multiplication xy of D. We have a nonassociative algebra $D^{(\lambda)}$ over Φ, called a *split quasi-associative algebra* if $\lambda \neq \frac{1}{2}$. A nonassociative algebra A over Φ is called a *quasi-associative algebra* if there exist a splitting field $\Omega \supset \Phi$, an associative algebra D over Ω and $\lambda \in \Omega$, $\lambda \neq \frac{1}{2}$ such that $A_\Omega = \Omega \otimes_\Phi A = D^{(\lambda)}$. The element $\delta = \Delta^2$ where $\Delta = 2\lambda - 1$ is called the *discriminant* of A and is uniquely determined by A if we agree to use $\delta = 1$ for the associative algebra A.

Let A be a nonsplit quasi-associative algebra with identity over a field Φ of characteristic $\neq 2$ and a particular square root Δ of its discriminant δ is not in Φ. Define an algebra D to be

$$D = A_\Omega^{(\mu)} \text{ where } \Omega = \Phi(\Delta),\ \mu = \frac{1}{2}(1+\Delta^{-1}).$$

Consider an automorphism $*$ of the quadratic field $\Omega = \Phi + \Phi\Delta$ defined by $\Delta^* = -\Delta$, extending to an automorphism $* \otimes 1$ of $A_\Omega = \Omega \otimes A$: $(\omega \otimes a)^{(*\otimes 1)} = \omega^* \otimes a$, $(\alpha + \beta\Delta)^* = \alpha - \beta\Delta$ for $\omega \in \Omega$, $a \in A$, $\alpha, \beta \in \Phi$. Let us denote $* \otimes 1$ by $*$ again. K. McCrimmon in [2] has shown that D is associative and

$$A = H(D^{(\lambda)}, *),\ \lambda = \frac{1}{2}(1+\Delta),$$

the subalgebra of symmetric elements of $D^{(\lambda)}$ under the $*$. Assuming that $\lambda \neq 0, 1, \frac{1}{2}$ we use this representation of A to determine an arbitrary automorphism σ of A. Since $(x \cdot y)^\sigma = x^\sigma \cdot y^\sigma$, $x, y \in A$ and σ is linear on the vector space of A, $(xy)^\sigma - x^\sigma y^\sigma = \lambda^{-1}(\lambda-1)\{(yx)^\sigma - y^\sigma x^\sigma\}$. By interchanging the role of x and y, $(yx)^\sigma - y^\sigma x^\sigma = \lambda^{-1}(\lambda-1)\{(xy)^\sigma - x^\sigma y^\sigma\}$ and hence $(xy)^\sigma - x^\sigma y^\sigma =$

$\{\lambda^{-1}\lambda-1\}^2\{(xy)^\sigma - x^\sigma y^\sigma\}$. It follows from the assumption $\lambda\neq\frac{1}{2}$ that $(xy)^\sigma - x^\sigma y^\sigma = 0$, that is, $(xy)^\sigma = x^\sigma y^\sigma$ for all x, y of A. Since $\Omega=\Phi(\Delta)=\Phi+\Phi\Delta$, $D^{(\lambda)}=A_\Omega=A+A\Delta$. We extend the σ to a linear transformation $1\otimes\sigma$ of A_Ω by defining $(a+b\Delta)^{(1\otimes\sigma)} = a^\sigma + b^\sigma\Delta$. Then the linear extension $1\otimes\sigma$ is an automorphism of the associative algebra D, which is denoted by σ again. It follows that $\sigma* (=(1\otimes\sigma)(*\otimes 1)) = *\sigma$. Conversely, an automorphism σ of D which commute with the involution $*$ induces an automorphism of A. This completes the following

THEOREM. *Let A be a nonsplit quasi-associative algebra with identity over a field Φ of characteristic$\neq 2$. Let $A=H(D^{(\lambda)}, *)$ be McCrimmon's representation of A. Then the group of automorphisms of A consists of all automorphisms of the associative algebra D which commute with involution $*$.*

REMARK 1. If A is a split quasi-associative algebra, that is, $A=D^{(\lambda)}$, $\lambda\neq\frac{1}{2}$, for some associative algebra D, then the automorphism group of A is precisely the group of automorphisms of D.

REMARK 2. Let a quasi-associative algebra A be finite dimensional central simple over Φ. It is known that the symmetrized algebra A^+ is a central simple Jordan algebra of type A_{II} over Φ and that, in the representation $A=H(D^{(\lambda)}, *)$, the Jordan algebra $H(D, *)$ of $*$-symmetric elements of the associative algebra D is just A^+, i.e. $A^+=H(D, *)$. Since powers of an element a in A are identical with those of a in A^+, the generic norms in A and A^+ are identical. We apply a theorem of N. Jacobson ([1], p. 191) to obtain the following characterization of norm-preserving linear transformations of A: The group of bijective linear transformations of A which preserve the generic norm N is the set of linear transformations η of the form: $x\eta=\gamma a^* x^\nu a$ where $\gamma\in\Phi$, $\gamma^m N(a^*a) = 1$, m=the degree of A, and either $\nu=1$ or ν is an anti-automorphism of D over $\Omega=\Phi(2\lambda-1)$.

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REFERENCES

[1] N. Jacobson, *Some groups of transformations defined by Jordan algebras* I, J. Reine Angew. Math. 201, 1959, 178—195.

[2] K. McCrimmon, *A note on quasi-associative algebras*, Proc. Amer. Math. Soc., 17, 1966, 1455—1459.