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ALMOST c-CONTINUOUS FUNCTIONS

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1. Introduction.

In the literature, there are many weakened forms of continuity, for example, weak continuity, almost continuity, θ-continuity, upper (lower) semi continuity, feebly continuity, sequential continuity, etc. Since in 1971, the concept of c-continuity is introduced by Gentry & Hoyle Ⅲ [3], many properties of the c-continuous functions has been investigated by some authors, ([4], [5]).

In this paper, the author introduce a new weakened form of continuity, which is weaker than any of c-continuity or almost continuity, and call it the almost c-continuity. And some interesting properties of the almost c-continuous functions are introduced in the following section. Among the various definitions of almost continuity such as those introduced by Hussain, Stalling, Singal, Frolik, … etc. ([1], [6], [8]), only the concept of Singal is adopted in this paper. Throughout this paper all spaces are assumed to be topological spaces, and the notations "—" (bar) and "Int" stand for the closure and interior operators respectively.

DEFINITION 1. (Singal [2])

A function $f: X \to Y$ of a space X into a space Y is called *almost continuous at a point* $x \in X$, if for each neighbourhood V of $f(x)$ in Y, there exists a neighbourhood U of x in X such that $f(U) \subset \mathrm{Int}\overline{V}$. And f is called *almost continuous* (*on* X) if it is almost continuous at every point of X.

In Definition 1, the term neighbourhood can be replaced by open neighbourhood [2].

DEFINITION 2. (Karl R. Gentry and Hughes B. Hoyle Ⅲ [3])

A function $f: X \to Y$ is called *c-continuous at* $x \in X$ if for each open neighbourhood V of $f(x)$ in Y having compact complement, there exists an open neighbourhood U of x in X such that $f(U) \subset V$. And f is called *c-continuous* (*on* X) if it is c-continuous at every point of X.

The following lemmas are useful characterizations of c-continuous, and almost continuous functions.

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LEMMA 1. (Singal [2])

A function $f: X \to Y$ is almost continuous iff

(i) *The inverse image of every regularly open subset of Y is open in X, or equivalently*

(ii) *The inverse image of every regularly closed subset of Y is closed in X.*

LEMMA 2. *A function $f: X \to Y$ is c-continuous iff*

(i) [3] *The inverse image of every open subset of Y having compact complement is open in X, or equivalently*

(ii) [4] *The inverse image of every closed compact subset of Y is closed in X.*

As is to be shown, the concept of almost c-continuity is inseminated from the definitions 1 and 2.

DEFINITION 3.

A function $f: X \to Y$ is called *almost c-continuous at* $x \in X$, if for every open neighbourhood V of $f(u)$ in Y having compact complement, there exists an open neighbourhood U of x in X such that $f(U) \subset \mathrm{Int}(\bar{V})$, and as usual, he function is called *almost c-continuous on* X if it is almost c-continuous at every point of X.

Evidently, all continuous functions, c-continuous functions, almost continuous functions are almost c-continuous, but the converse is not true in general as the following example shows.

EXAMPLE. Let $X=R$ have the usual topology and let Y be the set $[0, \infty)$ in R whose topology has the sets $[0, 1]$, $\{1\}$, (r, ∞) with $r>1$, as its basic open sets.

Define $f: X \to Y$ by $f(x)=2$ if $x>0$, $f(x)=1$ if $x=0$, $f(x)=0$ if $x<0$.

Then the function f is neither almost continuous, nor c-continuous. But it is almost c-continuous on X.

Proof. All open subsets of Y containing $f(0)$ are $[1, \infty)$, $\{1\}$, $[0, 1]$, and Y. Among them, only the sets Y and $[1, \infty)$ have compact complement. For the regularly open subset $[0, 1]$ of $Y, f^{-1}([0, 1]) = (-\infty, 0]$, which is not open in X. So f is not almost continuous by Lemma 1. And for the open set $[1, \infty)$ with compact complement, $f^{-1}([1, \infty)) = [0, \infty)$, which is not open in X. Thus f is not c-continuous because of Lemma 2. But, since $\mathrm{Int}\overline{[1, \infty)} = \mathrm{Int}[0, \infty) = Y$, we have $f(X) \subset \mathrm{Int}\overline{[1, \infty)}$, showing that f is almost c-continuous at $x=0$, so is on X.

2. Characterizations of almost c-continuous functions.

THEOREM 1. *For a function $f: X \to Y$, the followings are equivalent.*

(i) *f is almost c-continuous*

(ii) *The inverse image of every regularly open subset of Y having compact complement is open in X*

(iii) *The inverse image of every regularly closed compact subset of Y is closed in X*

(iv) *For each $x \in X$, and each regularly open subset V of Y containing $f(x)$ having compact complement, there exists an open subset U of X containing x such that $f(U) \subset V$*

(v) *For each $x \in X$, and each open subset V of Y containing $f(x)$ having compact complement, $f^{-1}(\mathrm{Int}\overline{V})$ is open in X.*

Proof. (i)$\Rightarrow$(ii). Let V be any regularly open subset of Y having compact complement and let $x \in f^{-1}(V)$.

Then $f(x) \in V$. Since f is almost c-continuous, There exists an open neighbourhood U of x in X such that $f(U) \subset \mathrm{Int}\overline{V} = V$, so $U \subset f^{-1}(V)$ showing that $f^{-1}(V)$ is open in X.

(ii)$\Rightarrow$(iii). Let F be any regularly closed compact subset of Y, then $Y-F$ is a regularly open subset of Y having compact complement. Thus we have $f^{-1}(Y-F) = X - f^{-1}(F)$ is open in X by (ii).

(iii)$\Rightarrow$(iv). Let $x \in X$ be given and let V be a regularly open subset of Y containing $f(x)$ having compact complement. Then $Y-V$ is regularly closed compact. Hence by (iii) $f^{-1}(Y-V) = X - f^{-1}(V)$ is closed in X. Moreover we have $x \in f^{-1}(V)$, so letting $U = f^{-1}(V)$ gives the result.

(iv)$\Rightarrow$(v). For any open subset V of Y, $\mathrm{Int}\overline{V}$ is regularly open. And since Y-$\mathrm{Int}\overline{V}$ is a closed subset of $Y-V$, we know that $Y-\mathrm{Int}\overline{V}$ is compact. The result comes directly by the same method as the case (i) implies (ii).

(v)$\Rightarrow$(i). Let $x \in X$ be given, V be any open neighbourhood of $f(x)$ in Y having compact complement then the set $U = f^{-1}(\mathrm{Int}\ \overline{V})$ is an open neighbourhood of x in X with the property $f(U) = ff^{-1}(\mathrm{Int}\overline{V}) \subset \mathrm{Int}\overline{V}$.

THEOREM 2. *Any restriction of an almost c-continuous function is also almost c-continuous.*

Proof. Let $f: X \to Y$ be almost c-continuous, A be an arbitrary subset of X, and let V be a regularly open subset of Y having compact complement. Then $f^{-1}(V)$ is open in X, and hence $(f|_A)^{-1}(V) = f^{-1}(V) \cap A$ is an open subset of A.

THEOREM 3. *If $f: X \to Y$ is continuous and $g: Y \to Z$ is almost c-continuous, then $g \circ f: X \to Z$ is almost c-continuous.*

Proof. Let V be a regularly open subset of Z having compact complement.

Then $g^{-1}(V)$ is open in Y hence $(g\circ f)^{-1}(V)=f^{-1}(g^{-1}(V))$ is open in X by the continuity of f.

THEOREM 4. *Let $f:X\to Y$ be surjective open, then $f:Y\to Z$ is almost c-continuous if $g\circ f:X\to Z$ is almost c-continuous.*

Proof. Let W be any regularly open subset of Z having compact complement, then $(g\circ f)^{-1}(W)$ is open in X that is $f^{-1}(g^{-1}(W))$ is open. Since f is surjective open, $f(f^{-1}(g^{-1}(W)))=g^{-1}(W)$ is open.

LEMMA. *Let $f:X\to Y$ be a function, $x\in X$. If there exists an open neighbourhood U of x in X such that $f|_U$ is almost c-continuous at x, then f is almost c-continuous at x.*

Proof. Let V be a regularly open subset of Y containing $f(x)$ having compact complement, then since $f|_U$ is almost c-continuous at x, there exists an open subset U_1 of X such that $x\in U_1\cap U$ and $f(U_1\cap U)\subset V$ but $U_1\cap U$ is an open neighbourhood of x in the whole space X.

THEOREM 5. *Let $\{U_\alpha|\alpha\in A\}$ be an open cover of X. If $f|_{U_\alpha}$ is almost c-continuous for each $\alpha\in A$. Then f is almost c-continuous on X.*

Proof. It is straightforward from the Lemma.

THEOREM 6. *Let $f:X\to Y$ be a function, and $X=A\cup B$ where A and B are closed. If $f|_A$, $f|_B$ are almost c-continuous, then f is almost c-continuous.*

Proof. Let F be a regularly closed compact subset of Y, then since both $f|_A$ and $f|_B$ are almost c-continuous, we have $(f|_A)^{-1}(F)$, $(f|_B)^{-1}(F)$ are closed in A and B respectively, so are in X. Hence we get $f^{-1}(F)=(f|_A)^{-1}(F)\cup(f|_B)^{-1}(F)$ is closed in X.

THEOREM 7. *If $f:X\to Y$ be a function and $X=A\cup B$, and if both $f|_A$ and $f|_B$ are almost c-continuous at a point $x\in A\cap B$, then f is almost c-continuous at x.*

Proof. Let V be a regularly open subset of Y containing $f(x)$ having compact complement. Then, since $x\in A\cap B$, and both $f|_A$ and $f|_B$ are almost c-continuous at x, there exist open sets U_1, U_2 in X such that $x\in U_1\cap A$ with $f(U_1\cap A)\subset V$ and $x\in U_2\cap B$ with $f(U_2\cap B)\subset V$. So we have $f(U_1\cap U_2)=f(A\cap U_1\cap U_2)\cup f(B\cap U_1\cap U_2)\subset f(A\cap U_1)\cup f(B\cap U_2)\subset V$. Now $U_1\cap U_2$ is the required open neighbourhood of x.

THEOREM 8. *Let $f:X\to Y$ be almost c-continuous and let Y be a locally compact Hausdorff space. Then f has closed graph.*

Proof. Let's denote the graph of f by $G(f)$. If $(x,y)\in X\times Y-G(f)$, then $f(x)\neq y$. Hence there exist disjoint open sets V_1 and V_2 containing y

and $f(x)$ respectively. Since Y is locally compact Hausdorff, there exists an open subset V of Y such that $y \in V \subset \bar{V} \subset V_1$, with $\bar{V}$ compact. Since $\bar{V} = \overline{\mathrm{Int}\bar{V}}$, $f^{-1}(\bar{V})$ is closed in X which does not contain x. So there exists an open subset U of $X - f^{-1}(\bar{V})$ containing x such that $f(U) \subset \mathrm{Int}\,\overline{Y - \bar{V}} = \mathrm{Int}(Y - \overline{\mathrm{Int}\bar{V}}) = Y - \overline{\mathrm{Int}\bar{V}} = Y - \bar{V}$. Thus we have found an open neighbourhood $U \times V$ of (x, y) such that $(U \times V) \cap G(f) = \phi$.

THEOREM 9. *Let $f: X \to Y$ be a function, X be compact. If the graph function $g: X \to X \times Y$ via $x \to (x, f(x))$ is almost c-continuous, then f is almost c-continuous.*

Proof. Let $x \in X$ and V be an open neighbourhood of $f(x)$ having compact complement, then $\pi_2^{-1}(V)$ is open in $X \times Y$. Since X and $Y - V$ are compact, $X \times (Y - V) = X \times Y - \pi_2^{-1}(V)$ is compact, so we know that $\pi_2^{-1}(V)$ is an open subset of $X \times Y$ having compact complement. Hence there is an open neighbourhood U of x in X such that $g(U) \subset \mathrm{Int}\overline{\pi_2^{-1}(V)} = \mathrm{Int}\bar{X} \times V = \mathrm{Int}\bar{X} \times \mathrm{Int}\bar{V} = X \times \mathrm{Int}\bar{V} = \pi_2^{-1}(\mathrm{Int}\bar{V})$. So we have $\pi_2 g(U) = f(U) \subset \pi_2 \pi_2^{-1}(\mathrm{Int}\bar{V}) \subset \mathrm{Int}\bar{V}$.

The following theorem gives a sufficient condition under which an almost c-continuous function becomes a continuous function.

THEOREM 10. *Let $f: X \to Y$ be almost c-continuous, X be of first countable, and let Y be a locally compact countable compact Hausdorff space. Then f is continuous.*

Proof. Suppose f is not continuous at a point x in X, then there is an open neighbourhood V of $f(x)$ in Y such that $f(U) \not\subset V$ for every open neighbourhood U of x in X. Let $U_1, U_2, \cdots$ be a countable base at x, and choose a point $x_n \in U_n$ such that $f(x_n) \notin V$ for each $n = 1, 2, \cdots$. Then x_n converges to x and the sequence $\langle f(x_n) \rangle$ has an accumulation point $y \in V$ in the countable compact space Y. By the Hausdorff property of Y we can take a pair V_1, V_2 of disjoint open sets such that $f(x) \in V_1 \subset V$, $y \in V_2$. Also there exists an open set W in Y such that $y \in W \subset \bar{W} \subset V$ with $\bar{W}$ being compact since Y is locally compact Hausdorff. Thus $Y - \bar{W}$ is an open neighbourhood of $f(x)$ having compact complement. But if U is any open neighbourhood of x, then there is a $U_n \subset U$ and a point $x_n \in U_n$ such that $f(x_n) \in W$ since $\langle f(x_n) \rangle$ converges to y. Hence $f(U) \not\subset Y - W$, so $f(U) \not\subset Y - \overline{W} = Y - \overline{\mathrm{Int}\,\bar{W}} = \mathrm{Int}\,\overline{Y - \bar{W}}$.

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