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SEQUENTIALLY CLOSED SPACES

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1. Introduction.

We have shown [15] that for any hereditary category $\boldsymbol{A}$ of Hausdorff (uniform) spaces and continuous (uniformly continuous, resp.) maps and any extensive subcategory $\boldsymbol{B}$ of $\boldsymbol{A}$, every reflective subcategory of $\boldsymbol{A}$ containing $\boldsymbol{B}$ is characterized by some extensive operator on $\boldsymbol{B}$, and that every extensive operator can be induced by some coreflective subcategory of the category **Top** of topological spaces and continuous maps. Franklin has shown [5, 6, 7] that the category $\boldsymbol{S}_{\omega_0}$ of sequential spaces and continuous maps is coreflective in **Top**. Herrlich has also shown [11] that the category $\boldsymbol{S}_\alpha$ of α-sequential spaces and continuous maps for any regular ordinal α is coreflective in **Top**. Using these results, we construct another chain of reflective subcategories in various categories of Hausdorff (uniform) spaces and continuous (uniformly continuous, resp.) maps and investigate properties of those categories.

All topological and categorical concepts will be used in the sense of N. Bourbaki [4] and Herrlich and Strecker [13], respectively. However, we assume throughout this paper that a subcategory of a category is full and isomorphism-closed.

2. α-sequential spaces.

2.1 DEFINITION. Let α be a regular ordinal. A net is said to be an *α-sequence* if its domain is the well-ordered set α, i.e. the set of all ordinals less than α.

2.2 DEFINITION. A subset U of a topological space X is *α-sequentially open* if each α-sequence in X converging to a point of U is eventually in U. A topological space X is called *α-sequential* if each α-sequentially open subset of X is open.

It is noted that ω_0-sequential spaces are precisely sequential spaces.

For a subset A of a topological space X and for a regular ordinal α, we define $l_X{}^\alpha A$ by $\{x\in X \mid$ there is an α-sequence in A converging to $x\}$. Then

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it is easy to show that the operator l^α defined as above is a limit-operator on **Top** (see [11]). Moreover, it is known [11] that l^α is not idempotent and that the category S_α of α-sequential spaces and continuous maps generates the associated idempotent limit-operator with l^α.

2.3 DEFINITION. A filter $\mathfrak{F}$ on a set X is called an *α-filter* for a regular ordinal α if it has a base $(B_\lambda)_{\lambda<\alpha}$ such that $B_\lambda \subseteq B_\mu$ for $\mu \leq \lambda < \alpha$.

It is again noted that ω_0-filters on a set are exactly filters with countable bases.

2.4 DEFINITION. A filter on a set X is said to be an *α-Fréchet filter* for a regular ordinal α if it is generated by the tails of an α-sequence on X.

2.5 PROPOSITION. *Every α-filter on a set X is the intersection of the α-Fréchet filters containing it.*

Proof. The proof is simple and left to the reader (also see Proposition 11, §6, Chap. 1 in [4]).

In the following, by an extension space of a space X is meant a space of which X is a dense subspace.

2.6 DEFINITION. Let Y be an extension space of a space X. For each point y of Y, $T(y)=\{V\cap X \mid V\text{: open neighborhood of } y \text{ in } Y\}$ will be called the *trace filter of y on X*. And the family of $(T(y))_{y\in Y}$ will be called the *filter trace of Y on X*.

2.7 PROPOSITION. *Let Y be an extension space of a space X. The following are equivalent for the limit-operator l^α:*

1) $l_Y{}^\alpha X = X$, *i.e. X is l^α-closed in Y.*

2) *For any $y\in Y$, if there is an α-Fréchet filter on X containing its trace filter $T(y)$, then $y\in X$.*

3) *For any $y\in Y$, if there is an α-filter containing its trace filter $T(y)$, then $y\in X$.*

Proof. 1)$\Rightarrow$2). Let $\mathfrak{F}$ be an α-Fréchet filter containing $T(y)$ and let $(x_\lambda)_{\lambda<\alpha}$ be an α-sequence in X which generates $\mathfrak{F}$. By the definition of the trace filter, it is obvious that the α-sequence (x_λ) converges to y. Hence y belongs to $l_Y{}^\alpha X = X$.

2)$\Rightarrow$3). It follows immediately from Proposition 2.5.

3)$\Rightarrow$1). For any $y\in l_Y{}^\alpha X$, there is an α-sequence $(x_\lambda)_{\lambda<\alpha}$ in X converging to y. It is obvious that $T(y)$ is contained in the α-Fréchet filter generated by the tails of (x_λ) ; $y\in X$.

3. α-sequentially closed spaces.

3.1 DEFINITION. Let $\boldsymbol{A}$ be a subcategory of the category **Haus** of Hausdorff spaces and continuous maps or **HUnif** of Hausdorff uniform spaces and uniformly continuous maps. A subcategory $\boldsymbol{B}$ of $\boldsymbol{A}$ is said to be *extensive* if it is a reflective subcategory of $\boldsymbol{A}$ such that the $\boldsymbol{B}$-reflection maps $r_X : X \to rX$ are dense embeddings for each $X \in \boldsymbol{A}$.

3.2 DEFINITION. A completely regular space X is called *β-α-sequentially closed* if every maximal completely regular filter on X which is contained in an α-filter is convergent.

It is well known that the category **Comp** of compact spaces and continuous maps is extensive in the category **CReg** of completely regular spaces and continuous maps and that a reflection of a completely regular space X is given by its Stone-Čech compactification βX. Recall that βX is the strict extension (see [3]) of X with all maximal completely regular filters on X as the filter trace (see [4]). Since $\bar{l}_{\beta X}{}^{\alpha} X = X$ iff X is l^{α}-closed in βX, where $\bar{l}^{\alpha}$ is the associated idempotent limit-operator with l^{α}, the following is immediate from Theorem 2.6 [15] (or see [14]) and Proposition 2.7.

3.3 THEOREM. *The category* $\mathbf{Comp}_{\alpha}$ *of β-α-sequentially closed spaces and continuous maps is extensive in the category* **Creg** *of completely regular spaces and continuous maps.*

3.4 COROLLARY. *The category* $\mathbf{Comp}_{\alpha}$ *is productive and closed-hereditary.*

3.5 DEFINITION. A Hausdorff space is called *zero-dimensional* if it has a basis consisting of sets which are both open and closed.

3.6 DEFINITION. A zero-dimensional space X is called *ζ-α-sequentially closed* if every maximal open closed filter on X which is contained in an α-filter is convergent.

It is known that the subcategory **ZComp** of all zero-dimensional compact spaces is extensive in the category **Zero** of zero-dimensional spaces and continuous maps and that the reflection of a zero-dimensional space X is given by the maximal zero-dimensional compactification ζX of X which is given by the strict extension of X with all maximal open closed filters on X as the filter trace (see [2] and [3]). The category of ζ-α-sequentially closed spaces and continuous maps will be denoted by $\mathbf{ZComp}_{\alpha}$. By the same argument as Theorem 3.3, we have the following:

3.7 THEOREM. *The category* $\mathbf{ZComp}_{\alpha}$ *is extensive in the category* **Zero**.

3.8 COROLLARY. *The category* $\mathbf{ZComp}_{\alpha}$ *is productive and closed hereditary.*

It is also well known [4] that the subcategory **Compl** of all complete Hausdorff uniform spaces is extensive in the category **HUnif** via the completions.

3.9 DEFINITION. A Hausdorff uniform space X is called *c–α–sequentially closed* if every Cauchy filter on X which is contained in an α–filter is convergent.

In this paper, the completion cX of a Hausdorff uniform space X is understood as in [4] and we identify each point of X with its neighborhood filter, so that X is a subspace of cX. Using the fact that each minimal Cauchy filter ξ is generated by $\{V(F) \mid V\text{: symmetric entourage on } X, F\in\xi\}$, it is easy to show that the trace filter of $\xi\in cX$ on X generates ξ itself. Moreover, for any Cauchy filter η, there is a unique minimal Cauchy filter which is coarser than η. Hence for any Hausdorff uniform space X, X is l^{α}–closed in cX iff it is c–α–sequentially closed. Thus we have the following:

3.10 THEOREM. *The category* $\mathbf{Compl}_{\alpha}$ *of c–α–sequentially closed spaces and uniformly continuous maps is extensive in* **HUnif**.

3.11 COROLLARY. *The category* $\mathbf{Compl}_{\alpha}$ *is productive and closed hereditary.*

3.12 DEFINITION. A Hausdorff space X is called *κ–α–sequentially closed* if every maximal open filter on X which is contained in an α–filter is convergent.

The subcategory of **pHaus** (see [9]) determined by all κ–α–seqentially closed spaces will be denoted by $\boldsymbol{A}_{\alpha}$.

Since the Katětov extension κX (see [17]) of a Hausdorff space X is the simple extension of X with all nonconvergent maximal open filters together with all open neighborhood filters on X as the filter trace (see [3]), X is l^{α}–closed in κX iff X is κ–α–sequentially closed. Thus we have the following by Remark 2.7, 2) [15].

3.13 THEOREM. *The category* $\boldsymbol{A}_{\alpha}$ *is extensive in the category* **pHaus**.

By the same argument, it is not difficult to show that the category of all κ–α–sequentially closed spaces and continuous semi–open maps is extensive in the category **Haus*** of Hausdorff spaces and continuous semi–open maps (see [12]). Hence we have the following by Proposition 2 and 3 in [12].

3.14 PROPOSITION. 1) *Every product of κ–α–sequentially closed spaces is also κ–α–sequentially closed.*

2) *Every regular closed subspace of a κ–α–sequentially closed space is also κ–α–sequentially closed.*

The following definition is due to Alexandroff and Urysohn [1].

3.15 DEFINTION. Let $\aleph$ be an infinite cardinnal number. A Hausdorff space X is called $\aleph$-$\aleph_0$ *compact* if every open covering $\mathcal{U}$ of X with $|\mathcal{U}| \leqslant \aleph$, has a finite subcovering.

We note that $\aleph_0$-$\aleph_0$ compact spaces are precisely countably compact spaces.

3.16 THEOREM. *Let α be a regular ordinal and let $\aleph$ be the cardinal number of α. Then every $\aleph$-$\aleph_0$ compact space is κ-α-sequentially closed, and every completely regular (zero-dimensional, Hausdorff uniform) $\aleph$-$\aleph_0$ compact space is $\beta(\zeta, c$, respectively$)$-α-sequentially closed.*

Proof. It is immediate from the fact that a Hausdorff space is $\aleph$-$\aleph_0$ compact iff every filter with a base whose cardinal number is not greater than $\aleph$ has a cluster point and the fact that every neighborhood filter in a completely regular (zero-dimensional, uniform) space is a maximal completely regular (maximal open closed, minimal Cauchy, respectively) filter.

3.17 REMARK. It is well known that every countably compact space is pseudo-compact. However, there is a pseudo-compact space namely $I_{\aleph^1} - \{p\}$ where I is the unit interval and all the coordinates of p are 1. But the space is not β-ω_0-sequentially closed, for $\beta(I_{\aleph^1} - \{p\}) = I_{\aleph^1}$ (see [16]).

3.18 THEOREM. *Every discrete space is $\kappa(\beta, \zeta, c)$-ω_0-sequentially closed.*

Proof. It is immediate from the fact that every ultrafilter on a discrete space which is a Fréchet filter is convergent.

3.19 EXAMPLE. The smallest ordinal of a cardinal number χ_λ is denoted by ω_λ. Let λ be a nonlimit ordinal>0 and $W(\omega_\lambda)$ the space of all ordinals less than ω_λ endowed by the interval topology. Then it is known [8] that no subset of $W(\omega_\lambda)$ of cardinal number $<\aleph_\lambda$ is cofinal and that $\beta W(\omega_\lambda) = \zeta W(\omega_\lambda) = W(\omega_\lambda + 1)$. Moreover, the Stone-Čech compactification βX of a completely regular space X is precisely the completion of X with the uniform structure generated by the set $C^*(X)$ of all real bounded continuous maps. Hence for any regular ordinal $\alpha < \omega_\lambda$, $W(\omega_\lambda) \in \mathbf{Comp}_\alpha$ $(\mathbf{ZComp}_\alpha, \mathbf{Compl}_\alpha)$. But for any regular ordinal $\alpha \geqslant \omega_\lambda$, $W(\omega_\lambda) \notin \mathbf{Comp}_\alpha$ $(\mathbf{ZComp}_\alpha, \mathbf{Compl}_\alpha)$.

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