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AN OPERATOR VALUED CONTINUOUS HOMOMORPHISM ON A B^*-ALGEBRA

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1. Introduction.

The main purpose of this note is to make the following integrals $\int_S f(s) E(ds), \int_{\sigma(\psi e)} f(s)E(ds)$ and $\int_\Delta (\psi f)^\wedge(h)E(dh)$ identify in some sense. It is shown that $\sigma(\psi e)\equiv S$ (theorem 2.7), $S\equiv\Delta$ (proposition 3.2) and an identification $\int_{\sigma(\psi e)} f(s)E(ds)\equiv\int_\Delta (\psi f)^\wedge(h)E(dh)$ is given (Corollary 3.3). It results that the above three integrals are identified.

Throughout this note we shall denote by $B(H)$ the algebra of all bounded linear operators defined on the complex Hilbert space H, let $C(S)$, $\hat{T}$, Δ be complex continuous functions defined on a compact subset S in $\boldsymbol{C}$, the Gelfand transform of an operator T and the maximal ideal space of the B^*-algebra respectively. Let Σ denote the σ-field of subsets of S, let $B(S,\Sigma)$ be consists of uniform limit of finite linear combinations of characteristic functions of sets in Σ.

2. An operator valued continuous homomorphism.

LEMMA 2.1. *Let $\psi: B(S,\Sigma)\rightarrow B(H)$ be a continuous algebraic homomorphism. If we define*

$$\psi(\chi_\delta)=(\psi\circ\chi)(\delta)=E(\delta) \quad \textit{for each } \delta\in\Sigma,$$

then the mapping $E:\Sigma\rightarrow B(H)$ defines a spectral measure with $\|E(\delta)\|=1$ for each non-empty $\delta\in\Sigma$.

Proof. Since

$$E^2(\delta)=E(\delta)\cdot E(\delta)=\psi(\chi_\delta)\cdot\psi(x_\delta)=\psi(\chi_\delta)$$
$$=E(\delta) \quad \text{for each } \delta\in\Sigma,$$

we have $E^2=E$.

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From the definition, it is obvious that

$$\psi(1)=I \quad 1(s)=1 \text{ for all } s\in S, \quad \|\psi(1)\|=1.$$

Moreover

$$E(\sigma\cap\delta)=\psi(\chi_{\sigma\cap\delta})=\psi(\chi_\sigma\cdot\chi_\delta)=(\psi\chi_\sigma)(\psi\chi_\delta)=E(\sigma)\wedge E(\delta),$$

$$\begin{aligned}E(\sigma\cup\delta)&=\psi(\chi_{\sigma\cup\delta})=\psi(\chi_\sigma+\chi_\delta-\chi_{\sigma\cap\delta})\\&=\psi(\chi_\sigma)+\psi(\chi_\delta)-\psi(\chi_{\sigma\cap\delta})\\&=E(\sigma)+E(\delta)-E(\sigma)\wedge E(\delta)\\&=E(\sigma)\vee E(\delta).\end{aligned}$$

Thus $E:\Sigma\to B(H)$ is a homomorphism on the Boolean algebra to a Boolean algebra of projection operators in $B(H)$.

Furthermore if $\{\sigma_i\}_{i=1}^{\infty}\subset\Sigma$ with $\sigma_i\cap\sigma_j=\phi$ $(i\neq j)$, then

$$\begin{aligned}E\left(\bigcup_{i=1}^{\infty}\sigma_i\right)x&=\psi(\chi_{\bigcup_{i=1}^{\infty}\sigma_i})x=\psi(\lim_{n\to\infty}\chi_{\bigcup_{i=1}^{n}\sigma_i})x\\&=\lim_{n\to\infty}\sum_{i=1}^{n}\psi(\chi_{\delta_i})x=\sum_{i=1}^{\infty}E(\sigma_i)x\end{aligned}$$

for any $x\in H$.

The series converges ([4], p. 295, Theorem 12.6).

If $\sigma\subseteq\delta$ for $\sigma,\delta\in\Sigma$, then $\delta=\sigma\cup(\delta\setminus\sigma)$

$$\chi_\delta=\chi_\sigma+\chi_{\delta\setminus\sigma} \text{ and } \psi(\chi_\delta)=\psi(\chi_\sigma)+\psi(\chi_{\delta\setminus\sigma}).$$

$$\text{i.e. } E(\delta)=E(\sigma)+E(\delta\setminus\sigma).$$

Hence

$$E(\sigma)H\subseteq E(\delta)H.$$

It follows that

$$\|E(\sigma)\|\leq\|E(\delta)\|.$$

Since $\delta\subseteq S$ for any $\delta\in\Sigma$,

$$\|E(\delta)\|\leq\|E(S)\|=\|I\|=1.$$

On the other hand $\|E(\delta)\|=\|E(\delta)^2\|\leq\|E(\delta)\|\ \|E(\delta)\|$

$$\therefore\ 1\leq\|E(\delta)\| \text{ for any non-empty } \delta\in\Sigma.$$

Thus we have

$$\|E(\delta)\|=1 \text{ for } \phi\neq\delta\in\Sigma. \tag{2.1}$$

LEMMA 2.2. *For a* $f\in B(S,\Sigma)$, ψf *is a spectral operator with the spectral resolution* $E=\psi\circ\chi$.

Proof. By the Lemma 2.1, it is enough to show that

$$E(\delta)\cdot(\psi f)=(\psi f)\cdot E(\delta) \text{ and } \sigma(\psi f|E(\delta)H)\subseteq\bar{\delta},\ \delta\in\Sigma.$$

The first equility is obvious since

$$(\psi f)E(\delta)=(\psi f)(\psi\chi_\delta)=\psi(f\chi_\delta)=\psi(\chi_\delta f)$$
$$=(\psi\chi_\delta)\cdot(\psi f)=E(\delta)(\psi f).$$

For $\lambda\notin\bar{\delta}$, $\chi_\delta(\lambda)=0$, whence

$$\psi(\chi_\delta(\lambda))H=\{0\}\ \text{(a zero vector)}.$$

Since $\sigma(\psi f|\{0\})=\phi$, it follows that

$$\lambda\notin\sigma(\psi f|(\psi\chi_\delta)H)\ \text{ if } \lambda\notin\bar{\delta}.$$
$$\therefore\sigma(\psi f|E(\delta)H)\subseteq\bar{\delta}.$$

For the resolution E, we have the following

LEMMA 2.3. *For a* $f\in B(S,\Sigma)$, ψf *can be represented in the form* $\psi f=\int_S f(s)E(ds)$.

Proof. For a Σ-simple function f on S is of a form

$$f=\sum_{i=1}^n\alpha_i\chi_{\delta_i}.$$

Hence we have

$$\psi f=\sum_{i=1}^n\alpha_iE(\delta_i). \tag{2.2}$$

In this case ψf is determinded uniquely since if

$$\sum_{i=1}^n\alpha_i\chi_{\sigma_i}=\sum_{j=i}^m\beta_j\chi_{\delta_j},\ \text{ then }\ \sum_{i=1}^n\alpha_iE(\sigma_i)=\sum_{j=1}^m\beta_jE(\delta_j).$$

For an arbitrary $f\in B(S,\Sigma)$ there exist a sequence of Σ-simple functions $\{f_n\}$ such that $f_n\to f$ uniformly on S. It follows from (2.2) that

$$\psi f_n\to\psi f=\int_S f(s)E(ds). \tag{2.3}$$

If $f=\chi_\delta$, then

$$\psi\chi_\delta=\int_S\chi_\delta E(ds)=\int_\delta E(ds)=E(\delta),$$

and for any Σ-simple function $f=\sum_{i=1}^{n}\alpha_i\chi_{\delta_i}$

$$\int_S f(s)E(ds)=\sum_{i=1}^{n}\alpha_i E(\delta_i)=\psi f.$$

Thus the formula (2.3) valid for any $f\in B(S,\Sigma)$.

Without proof we may give the following

PROPOSISION 2.4. *Suppose that $E(\delta)^*=E(\delta)$ for any $\delta\in\Sigma$, then $A=\{\psi f\in B(H):f\in B(S,\Sigma)\}$ is a commutative B^*-algebra, and the continuous homomorphism $\psi:B(S,\Sigma)\to A$ defines a *-homomorphism of B^*-algebra $B(S,\Sigma)$ onto B^*-algebra A.*

Thus A forms a B^*-algebra and the equality $(\psi f)^*=\int_S \bar{f}(s)E(ds)$ implies that the involution on A is mapped by ψ to the natural involution on $B(S,\Sigma)$.

THEOREM 2.5. *Let Δ be the maximal ideal space of the commutative B^*-algebra $A=\{\psi f:f\in B(S,\Sigma)\}$ with $(\psi(\chi_\delta))^*=\psi(\chi_\delta)$ for each $\delta\in\Sigma$. Then any $F\in C(\Delta)$ (the algebra of complex continuous functions on Δ) can be expressed in the form*

$$F=\int_S f(s)\hat{E}(ds) \quad and \quad \bar{F}=\int_S \bar{f}(s)\hat{E}(ds) \tag{2.4}$$

for some $f\in B(S,\Sigma)$, where $\Lambda:A\to\hat{A}$ is the Gelfand transform: Moreover an evaluation of F to a $h\in\Delta$ can be written in the form

$$F(h)=\int_S f(s)\mu_h(ds),\quad \bar{F}(h)=\int_S \bar{f}(s)\mu_h(ds)$$

with $$|F(h)|\le\|f\| \text{ for each } h\in\Delta,$$

where μ_h is a complex measure on Σ determind uniquely by $h\in\Delta$ such that $\mu_h(S)=1$ and f is unique in the sense that a.e. $[\mu_h]$.

Proof. For a Σ-simple function $f=\sum_{i=1}^{n}\alpha_i\chi_{\delta_i}$, since the Gelfand transform $\Lambda:\psi f\to(\psi f)^\wedge$ is a homomorphism, we have

$$(\psi f)^\wedge=\sum_{i=1}^{n}\alpha_i\hat{E}(\delta_i) \text{ or } h(\psi f)=\sum_{i=1}^{n}\alpha_i h(E(\delta_i)).$$

Thus we have

$$\Big(\int_S f(s)E(ds)\Big)^\wedge=\sum_{i=1}^{n}\alpha_i\hat{E}(\delta_i)=\int_S f(s)\hat{E}(ds), \tag{2.5}$$

where $\hat{E}(\delta_i)\equiv(E(\delta_i))^\wedge$. From the facts that every complex homomorphism on a Banach algebra is continuous and the set of all Σ-simple functions is dense in $B(S,\Sigma)$, it follows that

$$(2.6)\qquad (\int_S f(s)E(ds))^\wedge=\int_S f(s)\hat{E}(ds)$$

holds for any $f\in B(S,\Sigma)$.

Since A is a commutative B^*-algebra (proposition 2.4), the Gelfand transform $\Lambda:A\to\hat{A}\subset C(\Delta)$ is isometrically isomorphic of A onto $C(\Delta)$ with the property that $[(\psi f)^*]^\wedge=\overline{(\psi f)^\wedge}$ (Gelfand-Naimark theorem).

Hence any $F\in C(\Delta)$ can be represented in the form

$$(2.7)\qquad F=(\psi f)^\wedge \text{ with } \|F\|=\|(\psi f)^\wedge\|_\infty=\|\psi f\|.$$

It follows from (2.6) and (2.7) that

$$(2.8)\qquad \bar{F}=(\psi f)^\wedge=\int_S f(s)\hat{E}(ds).$$

Moreover that

$$\bar{F}=\overline{(\psi f)^\wedge}=[(\psi f)^*]^\wedge=\int_S \bar{f}(s)\hat{E}(ds).$$

From (2.5) and (2.6), we see that

$$F(h)=(\psi f)^\wedge(h)=\int_S f(s)\hat{E}(ds)(h) \text{ for each } h\in\Delta.$$

If we put $\hat{E}(\delta)(h)=\mu_h(\delta)$, then $\mu_h:\Sigma\to\boldsymbol{C}$ is a complex measure defined on Σ and is uniquely determined by h. The last statement follows from the fact that the Gelfand transform is isomorphism. Thus if $h_1\neq h_2$, then

$$\hat{E}(\delta)(h_1)=\mu_{h_1}(\delta)\neq\mu_{h_2}(\delta)=\hat{E}(\delta)(h_2) \text{ for any } \phi\neq\delta\in\Sigma.$$

Therefore we have

$$(2.9)\qquad F(h)=\int_S f(s)\mu_h(ds),\quad \bar{F}(h)=\int_S \bar{f}(s)\mu_h(ds).$$

Furthermore

$$|F(h)|\leq\sup_{s\in S}|f(s)|\,|\mu_h(S)|,$$

and since

$$\mu_h(S)=h[E(S)]=h[\psi(1)]=h(I)=1 \quad ([4], \text{ p. } 231,$$

proposition 10. 6)

$$\therefore \ |F(h)| \leq \sup_{s\in S}|f(s)| = \|f\| \text{ for each } h\in\Delta.$$

The formula (2. 9) says that, for a given $F\in C(\Delta)$ there corresponds a unique (in the sense that almost every where modulo μ_h for each $h\in\Delta$) $f\in B(S,\Sigma)$.

This completes the proof.

Now we shall discuss characteristics of the complex measures $\{\mu_h : h\in\Delta\}$ defined in the Theorem 2. 5.

PROPOSITION 2. 6. *The one parameter family of complex measures* $\{\mu_h : h\in\Delta\}$ *defined by* $\mu_h(\delta)=\hat{E}(\delta)(h)$ *for* $\delta\in\Sigma$ *and* $h\in\Delta$ *has following properties*:

(1) $$\mu_{\alpha h}(\delta)=\alpha\mu_h(\delta),\ \alpha\in \mathbf{C}$$

(2) $$\mu_{h_1+h_2}(\delta)=\mu_{h_1}(\delta)+\mu_{h_2}(\delta)$$

(3) $$\mu_h(\phi)=0,\ \mu_h(S)=1 \text{ for any } h\in\Delta \text{ and}$$

$$0\leq|\mu_h(\delta)|\leq 1 \text{ for each } \delta\in\Sigma \text{ and } h\in\Delta.$$

Proof. (1), (2) and the first part of (3) are obvious.

In order to prove the inequality $0\leq|\mu_h(\delta)|\leq 1$, we observe the following:

If A is a Banach algebra, it is known that for any $x\in A$ with $\|x\|<1$, $|\phi(x)|<1$ holds for every complex homomorphism ϕ on A.

Suppose that $\|x\|=1$ and $|\lambda|>1$ then $I-\lambda^{-1}x$ is invertible since $\|\lambda^{-1}x\|<1$, whence $\phi(I-\lambda^{-1}x)\neq 0$ ([4], p. 231, Theorem 10. 7), I is the identity of A. Thus

$$\Phi(I)\neq\lambda^{-1}\Phi(x), \text{ that is, } \Phi(x)\neq\lambda.$$

$$\therefore\ |\Phi(x)|\leq 1.$$

It follows from (2. 1) that $|h[E(\delta)]|=|\mu_h(\delta)|\leq 1$ for each $h\in\Delta$. We shall use this complex measure again in the last section.

THEOREM 2. 7. *Let* S *be a compact subset of* $\mathbf{C}$ *and* $\psi: B(S,\Sigma)\to B(H)$ *be a continuous homomorphism.*

Then $\sigma(\psi e)=S$, *where* $e: S\to S$ *is the diagonal function defined by* $e(s)=s$ *for each* $s\in S$.

Proof. For any $\lambda\in S$, $\lambda I-\psi e=\psi(\lambda 1-e)$ (1 is the function such that $1(s)=1$ for each $s\in S$).

Since $(\lambda 1-e)(\lambda)=0$, it follows that $(\lambda 1-e)^{-1}\notin B(S,\Sigma)$, and $(\lambda 1-e)^{-1}$

exist in $B(S,\Sigma)$ if and only if $(\lambda I-\psi e)^{-1}$ exist in $B(H)$ since

$$(\lambda I-\psi e)^{-1}=\psi[(\lambda 1-e)^{-1}]$$

$$\therefore\ \lambda\in\sigma(\psi e),\ \text{i.e. } S\subseteq\sigma(\psi e).$$

Conversely suppose that $\lambda\notin S$, then the function $\lambda 1-e$ is one-to-one and continuous on S with $(\lambda 1-e)(z)\neq 0$ for all $z\in S$. It follows that $(\lambda 1-e)^{-1}\in B(S,\Sigma)$ and that $\psi[(\lambda 1-e)^{-1}]=(\lambda I-\psi e)^{-1}\in B(H)$.
Hence

$$\lambda\notin\sigma(\psi e),\ \text{so we have } \sigma(\psi e)\subseteq S.$$

Therefore

$$\sigma(\psi e)=S.$$

From this fact, ψf can be represented in the form

$$\psi f=\int_{\sigma(\psi e)} f(s)E(ds)\ \text{ for each } f\in B(S,\Sigma) \tag{2.10}$$

and

$$\psi e=\int_{\sigma(\psi e)} sE(ds).$$

PROPOSITION 2.8. *Let $A(S)$ be the algebra of all holomorphic functions in some neighborhood of S. Then $(\psi f)^{-1}$ exist in $A(S)$ if and only if $f(s)\neq 0$ for each $s\in S$. In this case we have*

$$(\psi f)^{-1}=\int_{\sigma(\psi e)}\frac{1}{f(s)}E(ds) \tag{2.11}$$

$$(\psi e)^{-1}=\int_{\sigma(\psi e)} s^{-1}E(ds)\ \text{ if } 0\notin S.$$

Proof. By the norm $\|f\|=\sup_{s\in S}|f(s)|$, the Weierstrass theorem on uniformly convergent sequences of holomorphic functions ([1], p. 330 Theorem 15.8) shows that $A(S)$ is closed in $B(S,\Sigma)$, whence it is a Banach subalgebra of $B(S,\Sigma)$.

If $f(s)\neq 0$ for every $s\in S$, then obviously $\frac{1}{f}\in A(S)$. Thus

$$g=\frac{1}{f}\in A(S),\ f\cdot g=1.$$

It follows that

$$(\psi f)(\psi g)=I=(\psi g)(\psi f)$$

i. e.

$$(\psi g)=(\psi f)^{-1}\in B(H).$$

Conversely, suppose $f(\alpha)=0$ for some $\alpha\in S$. Then there exist an $h\in A(S)$ such that $f(s)=(s-\alpha)h(s)$ for each $s\in S$. Thus $f=(e-\alpha 1)h$, $\psi f=(\psi e-\alpha I)\psi h$. Since $S=\sigma(\psi e)$, $(\psi e-\alpha I)^{-1}$ does not exists. Therefore $(\psi f)^{-1}$ does not exist in $B(H)$. Since $A(S)\subset B(S,\Sigma)$, it follows from (2.10) that

$$(\psi f)^{-1}=\psi\left(\frac{1}{f}\right)=\int_{\sigma(\psi e)}\frac{1}{f(s)}E(ds).$$

Moreover since $e\in A(S)$, if $0\notin S=\sigma(\psi e)$, then $(\psi e)^{-1}$ exist in $B(H)$ with

$$(\psi e)^{-1}=\int_{\sigma(\psi e)}s^{-1}E(ds).$$

THEOREM 2.9. *Let f be a continuous function on S. Then we have*

$$\sigma(\psi f)=f(\sigma(\psi e)). \tag{2.12}$$

Proof. For an arbitrary but fixed $\lambda\in \boldsymbol{C}$, $\lambda\in\sigma(\psi f)$ if and only if $\lambda I-\psi f$ is not invertible in $B(H)$, thus $(\lambda I-\psi f)^{-1}=(\psi(\lambda-f))^{-1}=\psi((\lambda-f)^{-1})$ does not exists. Since $\lambda-f$ is a continuous function on S, $(\lambda-f)^{-1}\in B(S,\Sigma)$ if and only if $\lambda-f\neq 0$ on S. Therefore $(\lambda-f)^{-1}\notin B(S,\Sigma)$ if and only if there exist a $\zeta\in S$ such that $f(\zeta)=\lambda$. Hence $\lambda\in\sigma(\psi f)$ if and only if there exist a $\zeta\in S$ such that $\lambda=f(\zeta)$. It follows that

$$\sigma(\psi f)=\{f(\zeta):\zeta\in S\}.$$

Since $\sigma(\psi e)=S$, $\sigma(\psi f)=f(\sigma(\psi e))$.

The formula (2.12) is a kind of spectral mapping theorem, but the underlying assumptions are different from the spectral mapping theorem for an operator $f(T)=\frac{1}{2\pi i}\int_C f(\zeta)(\zeta I-T)^{-1}d\zeta$.

COROLLARY 2.10. *If f is a continuous function on S, then*

$$\|\psi f\|=\|f\|=\gamma_\sigma(\psi f),\quad \|\psi e\|=\gamma_\sigma(\psi e), \tag{2.13}$$

where $\gamma_\sigma(.)$ is the spectral radius of an operator.

Proof. Since $(\psi f)^\wedge(\Delta)=\sigma(\psi f)$ for each $f\in B(S,\Sigma)$,

$$\|(\psi f)^\wedge\|_\infty=\sup_{\lambda\in\sigma(\psi f)}|\lambda|$$

and

$$\|(\psi f)^{\wedge}\|_{\infty}=\|\psi f\| \text{ for any } \psi f\in A.$$

Therefore if f is continuous on S, then

$$\|\psi f\|=\sup_{\lambda\in\sigma(\psi f)}|\lambda|=\sup_{\zeta\in\sigma(\psi e)}|f(\zeta)|=\|f\|.$$

Since e is a continuous function on S, replacing f by e we have

$$\|\psi e\|=\|e\|=\sup_{\lambda\in\sigma(\psi e)}|e(\lambda)|=\sup_{\lambda\in\sigma(\psi e)}|\lambda|=\gamma_{\sigma}(\psi e).$$

3. Identifications.

Let $C(S)$ be a set of all complex continuous functions on the compact set S in $\boldsymbol{C}$, and $\psi:C(S)\rightarrow B(H)$ be the continuous homomorphism with $E(\delta)^*=E(\delta)$ for each $\delta\in\Sigma$, then it is easy to show that

$$A_C=\{\psi f\in A: f\in C(S)\}$$

(where $A=\{\psi f\in B(H): f\in B(S,\Sigma)\}$) forms a B^*-subalgebra of A.

THEOREM 3.1 *Let Δ be the maximal ideal space of A_C. Then the formula $Tf=(\psi f)^{\wedge}$, $f\in C(S)$, defines an isometric isomorphism $T=\Lambda\circ\psi$ of $C(S)$ onto $C(\Delta)$ with $T\bar{f}=\overline{(\psi f)^{\wedge}}$ and $\|T\|=\|\psi\|=1$.*

Proof. Since A_C is a commutative B^*-algebra, A_C is isometrically isomorphic to $\hat{A}_C$. Thus

$$\|F\|=\|(\psi f)^{\wedge}\|_{\infty}=\|\psi f\|,\quad F\in C(\Delta).$$

From the fact that $\|\psi f\|=\|f\|$ for each $f\in C(S)$ ((2.13)), it follows that

$$\|Tf\|=\|f\|=\|F\|. \tag{3.1}$$

Since

$$\|f\|=\sup_{s\in S}|f(s)|,\ \|f\|=0 \text{ implies that } f=0.$$

Hence $Tf=0$ if and only if $f=0$, thus T is one-to-one. It follows from (3.1) that T is an isometric isomorphism of $C(S)$ onto $C(\Delta)$ with $\|T\|=\|\psi\|=1$.

Moreover $T\bar{f}=(\psi\bar{f})^{\wedge}=((\psi f)^*)^{\wedge}=\overline{(\psi f)^{\wedge}}$ holds by the Gelfand-Naimark theorem.

PROPOSITION 3.2. *Let Δ be the maximal ideal space of A_C, then S can be identified with Δ.*

Proof. By the theorem 3.1, $C(S)\equiv C(\Delta)$. If we put $f=e$, then $Te=(\psi e)^\wedge$ and $(\psi e)(\Delta)=\sigma(\psi e)$. We consider a mapping $(\psi e)^\wedge:\Delta\to\sigma(\psi e)$. From the fact that $\sigma(\psi f)=f(\sigma(\psi e))$ (Theorem 2.9) we have

$$\sigma(\psi f)=f(\sigma(\psi e))=f((\psi e)^\wedge(\Delta))=(f\circ(\psi e)^\wedge)(\Delta). \tag{3.2}$$

Suppose that

$$(\psi e)^\wedge(h_1)=(\psi e)^\wedge(h_2),$$

then

$$(\psi f)^\wedge(h_1)=(f\circ(\psi e)^\wedge)(h_1)=f((\psi e)^\wedge(h_1))$$
$$=f((\psi e)^\wedge(h_2))=(\psi f)^\wedge(h_2)$$

for each $f\in C(S)$;
Therefore $h_1(\psi f)=h_2(\psi f)$ holds for each $\psi f\in A_C$.

i.e. $h_1=h_2$.

It follows that the mapping $(\psi e)^\wedge:\Delta\to\sigma(\psi e)$ is one-to-one, onto and continuous. Since Δ is compact (with respect to the Gelfand topology), it follows that $(\psi e)^\wedge$ is homeomorphism of Δ onto $\sigma(\psi e)$.

The mapping $(\psi e)^\wedge$ also preserves complex conjugation; for, since

$$(\psi e)^\wedge(h)=\lambda_h\in\sigma(\psi e),\quad h\in\Delta,$$

we have

$$(\psi e)^\wedge(\bar{h})=\bar{h}(\psi e)=\overline{h(\psi e)}=\overline{(\psi e)^\wedge(h)}=\bar{\lambda}_h.$$

Thus

$$\Delta\equiv\sigma(\psi e)=S$$

COROLLARY 3.3. *If $f\in C(S)$, then the integral formula $\psi f=\int_{\sigma(\psi e)}f(s)E(ds)$ can be identified with*

$$\int_\Delta(\psi f)^\wedge(h)E(dh),$$

where Δ is the maximal ideal space of A_C.

Proof. From the Theorem 3.1 and the the proposition 3.2,

$$C(S)=C(\sigma(\psi e))\equiv C(\Delta)=\hat{A}_C,\quad \sigma(\psi e)\equiv\Delta.$$

Therefore

$$\int_{\sigma(\psi e)} f(s)\,E(ds) \equiv \int_{\Delta} (\psi f)^{\wedge}(h)\,E(dh) \text{ holds.}$$

Since

$$\sigma(\psi f) = (\psi f)^{\wedge}(\Delta) = (f \circ (\psi e)^{\wedge})(\Delta) \tag{3.2}$$

it follows that

$$\psi f = \int_{\Delta} (\psi f)^{\wedge}(h)\,E(dh) = \int_{\Delta} (f \circ (\psi e)^{\wedge})(h)\,E(dh). \tag{3.3}$$

If we write $E(dh) = dE(h)$ then

$$\psi f = \int_{\Delta} (\psi f)^{\wedge} dE \equiv \int_{\sigma(\psi e)} f\, dE.$$

THEOREM 3.4. *Let Δ be the maximal ideal space of A_C and $C^*(\Delta)$ be the dual space of $C(\Delta) \equiv C(\sigma(\psi e))$. Then we have the followings:*

(1) *The set of all complex measures $\{\mu_h : h \in \Delta\}$, determined in the section 2, can be regarded as a closed subspace of $C^*(\Delta)$;*

(2) *Δ can be embedded in $C^*(\Delta)$.*

Proof. We write $F(h) = (\psi f)^{\wedge}(h) = \int_S f d\mu_h$ in (2.9) for the convenience in future discussions. Fix $h \in \Delta$, a linear functional L_h is defined by

$$L_h(f) = (\psi f)^{\wedge}(h) = \int_{\sigma(\psi e)} f d\mu_h, \quad f \in C(\sigma(\psi e)). \tag{3.4}$$

The mapping $h \to L_h$ of Δ onto $\{L_h : h \in \Delta\} \subset C^*(\Delta)$ is one-to-one;
For, if $L_{h_1} = L_{h_2}$, then $(\psi f)^{\wedge}(h_1) = (\psi f)^{\wedge}(h_2)$ holds for every $f \in C(\sigma(\psi e))$. Thus $h_1 = h_2$. Therefore two sets $\{L_h : h \in \Delta\}$ and $\{\mu_h : h \in \Delta\}$ are identified. Therefore we may write

$$\mu_h(f) = \int_{\sigma(\psi e)} f d\mu_h, \quad \{\mu_h : h \in \Delta\} \subset C^*(\Delta). \tag{3.5}$$

We shall show that if $h_n \to h_0$ in the Gelfand topology of Δ, then $\mu_{h_n} \to \mu_{h_0}$ in the weak* topology of $C^*(\Delta)$.

Now, we recall that the Gelfand topology of Δ, it is a weak topology induced by $\hat{A}_C$. An ε-neighborhood $U(h_0)$ of $h_0 \in \Delta$ is following:

$$\begin{aligned} U(h_0) &= U(h_0 : (\psi f_1)^{\wedge}, \ (\psi f_2)^{\wedge}, \cdots, (\psi f_n)^{\wedge}, \varepsilon) \\ &= \{h \in \Delta : |(\psi f_i)^{\wedge}(h) - (\psi f_i)^{\wedge}(h_0)| < \varepsilon, \ i = 1, 2, \cdots, n\} \end{aligned}$$

for a finite positive integer n.
From (3.4) and (3.5) we have

$$(\psi f)^{\wedge}(h)=\mu_h(f) \text{ or } \langle h, (\psi f)^{\wedge}\rangle=\langle f, \mu_h\rangle.$$

Since

$$\hat{A}_C=C(\varDelta)=C(\sigma(\psi e)),$$

we may write

$$\langle h, f\rangle=\langle f, \mu_h\rangle. \tag{3.6}$$

Therefore a neighborhood of μ_{h_0} can be written in the form

$$V(\mu_{h_0})=\{\mu_h: |\mu_h(f_i)-\mu_{h_0}(f_i)|<\varepsilon, \ i=1,2,\cdots,n\}.$$

Hence $h\in U(h_0)$ if and only if $\mu_h\in V(\mu_{h_0})$. This implies that $h_n\to h_0$ in $\varDelta$ if and only if $\mu_{h_n}\to\mu_{h_0}$ in the weak* topology of $C^*(\varDelta)$. Since $h_0\in\varDelta$, $\mu_{h_0}\in C^*(\varDelta)$. It follows that $\{\mu_h: h\in\varDelta\}$ can be regarded as a closed subspace of $C^*(\varDelta)$.

From the proposition 2.6 together with the above facts, the mapping ϕ: $\varDelta\to\{\mu_h: h\in\varDelta\}$, defined by $\phi(h)=\mu_h$, is complex linear, one-to-one and bicontinuous. Thus ϕ is a homeomorphism, so we can identify $\varDelta$ with $\{\mu_h: h\in\varDelta\}\subset C^*(\varDelta)$. This completes the proof.

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