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# ON GEODESICS IN THE TANGENT SPACE OF A FINSLER SPACE

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Let $F^n$ be an $n$-dimensional Finsler space with the fundamental function $L(x,y)$ $(y^i=\dot{x}^i)$ and the fundamental tensor $g_{ij}(x,y)=(1/2)\dot{\partial}_i\dot{\partial}_j L^2(x,y)$. The tangent vector space $F_x{}^n$ with the origin removed at every point $x$ of $F^n$ is, of course, a Minkowski space with the norm $L(x,y)$. On the other hand, $F_x{}^n$ is also regarded as a Riemannian space with the fundamental quadratic form $ds^2=g_{ij}(x,y)dy^idy^j$ [8][1], as it is often emphasized in the monograph [4]. (See, for instance, Remark of p. 84 and §31.) The components $C_j{}^i{}_k$ of the $(h)hv$-torsion tensor of $F^n$ (Cartan's torsion tensor) are nothing but the connection coefficients of the Riemannian connection of $F_x{}^n$ and the $v$-curvature tensor $S_i{}^h{}_{jk}$ of $F^n$ (Cartan's first curvature tensor) is the Riemannian curvature tensor of $F_x{}^n$. Therefore the concept of geodesic is introduced in the Riemannian space $F_x{}^n$ by applying to $F_x{}^n$ the usual theory of calculus of variations, and a geodesic coincides with an autoparallel curve with respect to the Riemannian connection.

The Riemannian structure of $F_x{}^n$ is of distinct character, which is due to the structure of vector space and to the existence of fundamental function. Thus $C_j{}^i{}_k$ and $S_i{}^h{}_{jk}$ have certain conspicuous properties, as it is well-known in Finsler geometry, and it is naturally expected that the geodesics in $F_x{}^n$ must behave in some interesting manner. We have, however, only two papers [3] and [2] concerned with those geodesics, so far as the present author knows, and even these papers are not devoted to the studies of geodesics as the ones in the Riemannian space. (See §39 of [4].)

The purpose of the present paper is to show various results on those geodesics and the Jacobi fields along the geodesics. We will find the eccentric behavior on special geodesic, called the ray. In case of two dimensions the problem is of extremely simple character and becomes perfectly clear.

The results of the monograph [4] are often quoted in the following without too much comment. The quotation is denoted by putting the asterisk.

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1) Numbers in brackets refer to the references at the end of the paper.

## § 1. Preliminaries

We suppose throughout the paper that the fundamental function $L(x, y)$ of $F^n$ is positive-valued, positive-homogeneous of degree one, differentiable for any non-zero $y=(y^i)$ and $L(x, 0)=0$, and further the quadratic differential form

$$ds^2=g_{ij}(x, y)\,dy^i dy^j$$

is positive-definite.

Denote by $M_x{}^n$ the tangent vector space $F_x{}^n$ with the origin removed at a point $x$ of $F^n$. Then $M_x{}^n$ is thought of as an $n$-dimensional Riemannian space with the fundamental form $ds^2$. Because of the equations *(17.1) and *(17.1′) the components $C_j{}^i{}_k(x, y)$ of the $(h)hv$-torsion tensor $C$ (*§10) of the Cartan connection $C\Gamma$ of $F^n$ are the connection coefficients of the Riemannian connection of $M_x{}^n$ equipped with the above Riemannian structure. Further the $v$-curvature tensor $S$ of $C\Gamma$ (*§10) is nothing but the Riemannian curvature tensor of $M_x{}^n$, as it is seen from *(10.7′) and *(17.20).

Throughout the paper we are concerned with $M_x{}^n$ at the only one point $x$ of $F^n$ and do not write $x$ explicitly for all the quantities such as $L(y)$, $g_{ij}(y)$ and $C_j{}^i{}_k(y)$.

The concept of geodesic in this Riemannian space $M_x{}^n$ is first introduced by the present author in 1964 [3], called the $v$-path. (See *Definition 39.1 and *(39.6).) It is given by the differential equations

$$\frac{d^2y^i}{ds^2}+C_j{}^i{}_k(y)\frac{dy^j}{ds}\frac{dy^k}{ds}=0. \tag{1.1}$$

In the Riemannian space $M_x{}^n$ we are familiar with the concept of covariant differentiation $D_s$ along a curve $C=y(s)$. For a Finsler tensor field $T$ of $F^n$ it is written in the form

$$D_sT=\Delta^v T\left(\frac{dy}{ds}\right), \tag{1.2}$$

where $\Delta^v$ is the $v$-covariant differentiation with respect to the Cartan connection $C\Gamma$ of $F^n$ (*Definition 9.5) and is regarded as the usual covariant differentiation in $M_x{}^n$. For a Finsler scalar field $\alpha$ it is given by $D_s\alpha=\alpha'$, where and in the following the prime indicates $d/ds$. In particular $D_sy=y'$; it is denoted by $y_s$. Then we have for a Finsler vector field $U$

$$D_sU=U'+C(U, y_s), \tag{1.3}$$

where the notation $C(U, V)$ for two vectors $U, V$ is used to denote the vector with the components $C_j{}^i{}_kU^jV^k$. As a consequence the equations (1.1) are written in the simple form

(1.1') $$D_s y_s = 0.$$

The concept of Jacobi field along a geodesic $C=y(s)$ plays the essential role in the theory of geodesics; it is a vector field $J$ satisfying the so-called Jacobi differential equations

(1.4) $$D_s^2 J + S(y_s, y_s, J) = 0,$$

where $S(U, V, W)$ for three vectors $U, V, W$ stands for the vector with the components $S_h{}^i{}_{jk} U^h V^j W^k$. On account of the well-known expression

$$S_h{}^i{}_{jk} = C_h{}^r{}_k C_r{}^i{}_j - C_h{}^r{}_j C_r{}^i{}_k$$

of the $v$-curvature tensor $S$, the equation (1.4) are also written in the form

(1.4') $$D_s^2 J + C(y_s, C(y_s, J)) - C(J, C(y_s, y_s)) = 0.$$

Next we use the notation $\langle U, V\rangle_y$ to indicate the inner product $g_{ij}(y) U^i V^j$ of two vectors $U, V$. It then follows from the equation $\dot{\partial}_k g_{ij} = 2C_{ijk}$ or the metrical property of $\Delta^v$ that

(1.5) $$(\langle U, V\rangle)' = \langle D_s U, V\rangle + \langle U, D_s V\rangle$$

for two vector fields $U, V$ along a curve $C=y(s)$. As in (1.5), we do not write the subscript $y$ to show the point where the inner product is evaluated, in case there is no danger of confusion. Since $s$ is the arc length of the curve $C$, we obtain

(1.6) $$\langle y_s, y_s\rangle = 1.$$

It is, however, remarked that (1.6) does not mean $L^2(y_s) = g_{ij}(y_s) y_s^i y_s^j = \langle y_s, y_s\rangle_{y_s} = 1$, but $g_{ij}(y) y_s^i y_s^j = \langle y_s, y_s\rangle_y = 1$.

## §2. On the rays

We first pay attention to the vector space character of $M_x^n$. $y(s)$ of a curve $C=y(s)$ shows a point of $C$ and also the radius vector of the point. Then the value $L(y(s))$ is called the *length of the radius vector*.

The equation $L^2(y) = \langle y, y\rangle_y$ will be naturally understandable by $\langle y, y\rangle_y = g_{ij}(y) y^i y^j = L^2(y)$, but it is to be remarked that in the notation $\langle y, y\rangle_y$ we ragard $y$ as the tangent vector at the point $y$ of $M_x^n$ (*Example 2.2).

For a geodesic $C=y(s)$ it is observed from (1.1') and (1.6) that

$$(L^2(y))' = 2\langle y, y_s\rangle, \quad (L^2(y))'' = 2,$$

which implies

$$L^2(y(s)) = (s-c)^2 + c_0, \quad \langle y, y_s\rangle = s - c,$$

where $c$ and $c_0$ are some constants. Since $ds^2$ is supposed to be positive-definite, we have the notion of angle $y^\wedge y_s$ between the radius vector $y(s)$ and the tangent vector $y_s$ of $C$. It follows from the above and (1.6) that

$$\cos(y^\wedge y_s)=\frac{\langle y, y_s\rangle}{L(y(s))}=\frac{s-c}{\sqrt{(s-c)^2+c_0}}.$$

Hence the constant $c_0$ must be non-negative. Consequently we have

PROPOSITION 1. *The length $L(y(s))$ of the radius vector of a geodesic $C=y(s)$ in $M_x^n$ is written as*

(1) $$L^2(y(s))=(s-c)^2+h^2$$

*in terms of the arc length $s$ of $C$, where $c$ and $h$ $(\geq 0)$ are constants. Further we have*

(2) $$\langle y, y_s\rangle=s-c,$$

(3) $$\cos(y^\wedge y_s)=\frac{s-c}{\sqrt{(s-c)^2+h^2}}$$

The non-negative constant $h$ is called the ***height*** of the geodesic. Since a geodesic is uniquely determined by giving the initial point $y^0$ and the initial direction $y_s^0$ at $y^0$, we choose $y^0$ as $L(y^0)=h$ for a given positive $h$ and $y_s^0$ orthogonal to $y^0$ to obtain the geodesic with the height $h$.

REMARK. Although the inequality $L(y(s))\geq h$ holds good for any geodesic, it can not be stated that $L(y(s))$ reaches the minimum $h$. In fact, we should be careful about the fact that the origin is removed in $M_x^n$, and it may be possible $M_x^n$ is not complete. We shall soon show that $M_x^n$ is really not complete, hence it is not sure that any geodesic can be extendable to the point $s=c$ giving the minimum to $L(y(s))$.

DEFINITION. If the radius vector $y(s)$ of a curve $C=y(s)$ is positive-proportional to a non-zero constant vector $y_0$, then $C$ is called a *ray*.

Hence there exists for the ray $C=y(s)$ a positive-valued function $p(s)$ such as $y(s)=p(s)y_0$. The equation (1.6) implies

$$1=(p')^2\langle y_0, y_0\rangle_y=(p')^2g_{ij}(py_0)y_0^iy_0^j=(p')^2L^2(y_0),$$

because of the special condition of the metric such that $g_{ij}(y)$ is positive-homogeneous of degree zero. The non-zero constant vector $y_0$ may be assumed to be unit $(L(y_0)=1)$ and we obtain $p=s+c_1$ ($c_1$ being a constant) by suitable choice of the orientation of $s$. Conseqeuntly we obtain the equation of the ray:

(2.1) $$y(s)=(s+c_1)y_0, \quad (L(y_0)=1).$$

LEMMA 1. *A curve $C=y(s)$ is a ray, iff the radius vector $y(s)$ is proportional to the tangent vector $y_s$.*

*Proof.* The necessity of the condition is obvious from (2.1). Suppose conversely $y(s)$ be proportional to $y_s$. Then we may put $y_s=y(s)/L(y(s))$ and the integration gives $|y^i(s)|=c^i\exp(\int L^{-1}ds)$ with $n$ non-negative constants $c^i$. Putting $y_0{}^i=\pm c^i$, we conclude the positive-proportionality of $y(s)$ to $y_0$.

LEMMA 2. *A geodesic $C=y(s)$ has the height $h=0$, iff $y(s)$ is proportional to the tangent vector $y_s$.*

This will be clear from (3) of Proposition 1. Therefore a geodesic is a ray, iff the height $h$ vanishes. In fact any ray is a geodesic, because we see from (2.1) $y''=0$ and

$$C_j{}^i{}_k(y(s))y_s{}^k=\frac{1}{s+c_1}C_j{}^i{}_k(y_0)y_0{}^k=0.$$

Next, paying attention to the absence of the origin in $M_x{}^n$, we introduce

DEFINITION. A curve $C$ of $M_x{}^n$ is called *abnormal*, if $C$ has a Cauchy point sequence which converges to the origin of $F_x{}^n$; otherwise $C$ is called *normal*.

From (2.1) it is obvious that any ray is abnormal. We treat an abnormal geodesic $C=y(s)$. Since $L(0)=0$ is supposed and $L(y)$ is continuous at $y=0$, Proposition 1 shows that the height $h$ of $C$ must be equal to zero and $C$ is a ray.

We sum up the above results in the following:

THEOREM 1. (1) *A ray is an abnormal geodesic and the inverse is true.* (2) *A geodesic is a ray, iff the radius vector is proportional to the tangent vector or the height vanishes.*

Since a ray is an abnormal geodesic, it can not be infinitely extendable to both sides. Then, according to the well-known theorem on the completeness of Riemannian space, we conclude

COROLLARY. *The Riemannian space $M_x{}^n$, the tangent space of a Finsler space $F^n$ with the origin removed at a point $x$, is not complete.*

REMARK. A theorem due to K. Nomizu and H. Ozeki [5] asserts the existence of a complete Riemannian metric which is conformal to the given

Riemannian metric. But we are not interested in the theorem because of M. S. Knebelman's theorem [6, p. 224]: The conformal factor of any conformal transformation of a Finsler metric must be a function of position $x$ alone.

## § 3. On complete geodesics.

It has been seen that a ray (abnormal geodesic) has the origin as a limit point and can not be extendable over it. The purpose of the present section is to show the following Theorem 2. We first introduce

DEFINITION. If a normal geodesic $C$ in $M_x^n$ is infinitely extendable to both sides, then $C$ is called *complete*.

THEOREM 2. *In two-dimensional case all the normal geodesics in $M_x^2$ are complete.*

For a complete geodesic $C=y(s)$ the length $L(y(s))$ of the radius vector $y(s)$ reaches the minimum $h$ (height) at the point $s=c$ in the notations of Proposition 1. This point is called the *foot* of $C$. Then we take the foot as the initial point of the arc length $s$ to obtain

$$L^2(y(s))=s^2+h^2, \quad \langle y, y_s\rangle=s. \tag{3.1}$$

To prove Theorem 2, we refer to the Berwald frame $(1, m)$ and the Landsberg angle $\theta$ of the space $F^2$ (* § 28). The pair $(L, \theta)$ is regarded as a coordinate of a point of $M_x^2$. Let us first write the equations (1.1′) of a geodesic in terms of this coordinate. By the equation *(28.5) we have

$$y_s=\frac{L'}{L}y+L\theta' m. \tag{3.2}$$

On the other hand the equations *(28.4) and *(28.2) yield

$$\theta'=\langle m, y_s\rangle/L, \quad D_s m=-\frac{\theta'}{L}y.$$

Hence we obtain from (3.2)

$$D_s y_s=[(L'/L)'+(L'/L)^2-(\theta')^2]y+[(L\theta')'+L'\theta']m. \tag{3.3}$$

Accordingly the equations of geodesic are written in the form

$$(3.4) \qquad \begin{aligned} &(1) \quad (L'/L)'+(L'/L)^2-(\theta')^2=0,\\ &(2) \quad (L\theta')'+L'\theta'=0. \end{aligned}$$

Now we integrate these equations. It follows first from (2) that $L^2\theta'=a_0$ (constant). Then, in virtue of the equation in Proposition 1, the equation

(1) of (3.4) is reduced to $a_0^2=h^2$, so that we may put $a_0=h$ by suitable choice of the orientation of $\theta$. Thus we establish

THEOREM 3. *Any normal geodesic of a two-dimensional $M_x^2$ is given by the parametric equations*

$$L=\sqrt{(s-c)^2+h^2}, \quad \theta=\mathrm{Arctan}\frac{s-c}{h}+\alpha$$

*in terms of the length $L$ of the radius vector and the Landsberg angle $\theta$, where $c$, $h$ ($>0$) and $\alpha$ are constants.*

Now Theorem 2 is a corollary of Theorem 3. In fact the equations in Theorem 3 are defined for all real number $s$ and satisfy the equations (3.4) of geodesic.

Remember the parametric equations of a straight line in a euclidean plane in terms of polar coordinate. Then it is noticed that the equations in Theorem 3 is quite reasonable, as $(L, \theta)$ is regarded as a generalization of polar coordinate in a euclidean plane (cf. *Remark of p. 145).

We pay attention to the well-known fact (*Poposition 28.3) that the $v$-curvature tensor $S$ vanishes identically in any two-dimensional Finsler space. Therefore the sectional curvature of the Riemannian space $M_x^2$ vanishes, hence the well-known theorem due to I. J. Schoenberg [7] leads us immediately to

THEOREM 4. *Any normal geodesic of a two-dimensional $M_x^2$ has no conjugate point and is the relative shortest (minimal) geodesic.*

See the remark at the end of §6.

## §4. On geodesic surfaces

In this section the consideration is restricted to the Riemannian space $M_x^n$ of dimension $n$ more than two, and we are concerned with special two-dimensional subspaces of $M_x^n$:

DEFINITION. Let $C=y(s)$ be a normal curve in $M_x^n$ $(n>2)$. The surface $S$ defined by the equation $z(s,t)=ty(s)$ with the parameters $(s,t)$ is called the *surface generated by* $C$, where the parameter $t$ is supposed to be positive. If $C$ is a normal geodesic, then $S$ is called the *geodesic surface.*

Thus the surface $S$ generated by $C$ is a conical surface with the origin (although it is removed) as the vertex and the ray through every point of $C$ is the generator of $S$.

We consider a geodesic surface $S$ generated by a normal geodesic $C=y(s)$. Denoting the parameters $(s,t)$ of $S$ as $(t^\alpha)$ $(\alpha=1,2)$, we have two tangent vectors $Z_\alpha$ of $S$:

$$Z_1=\partial z/\partial t^1=ty_s,\quad Z_2=\partial z/\partial t^2=y.$$

By Theorem 1 it is seen that $Z_1$ and $Z_2$ are linearly independent. By (1.6) and Proposition 1 the fundamental tensor $g_{\alpha\beta}=g_{ij}Z_\alpha{}^iZ_\beta{}^j$ induced on $S$ is given by

$$g_{11}=t^2,\quad g_{12}=t(s-c),\quad g_{22}=(s-c)^2+h^2.$$

The $\det(g_{\alpha\beta})=(th)^2$; it does not vanish by Theorem 1, and $S$ has the positive-definite metric. The reciprocal $g^{\alpha\beta}$ of $g_{\alpha\beta}$ is given by

$$g^{11}=\frac{(s-c)^2+h^2}{(th)^2},\quad g^{12}=-\frac{s-c}{th^2},\quad g^{22}=\frac{1}{h^2}.$$

It is easy to show that the surviving Christoffel symbols $\Gamma_{\alpha\beta\gamma}$ and $\Gamma_\alpha{}^\beta{}_\gamma$ of $S$ are $\Gamma_{112}=t, \Gamma_{122}=s-c$ and $\Gamma_1{}^1{}_2=1/t$ alone.

The second fundamental tensors $H_{\alpha\beta}{}^P$ $(P=3,\cdots,n)$ of $S$ with respect to $n-2$ orthonormal vectors $N_P$ to $S$ are given by the Gauss formula

$$\frac{\partial Z_\alpha}{\partial t^\beta}+C(Z_\alpha, Z_\beta)-\Gamma_\alpha{}^\gamma{}_\beta Z_\gamma=H_{\alpha\beta}{}^P N_P.$$

By means of (1.1), the homogeneity of $C_j{}^i{}_k(y)$ and the equation $C_j{}^i{}_k(y)y^k=0$ it is observed that

$$H_{11}{}^P N_P{}^i=t\frac{d^2y^i}{ds^2}+C_j{}^i{}_k(ty)t^2y_s{}^jy_s{}^k=0,$$

$$H_{12}{}^P N_P{}^i=y_s{}^i+C_j{}^i{}_k(ty)ty_s{}^jy^k-\frac{1}{t}ty_s{}^i=0,$$

$$H_{22}{}^P N_P{}^i=C_j{}^i{}_k(ty)y^jy^k=0.$$

Thus all the $H_{\alpha\beta}{}^P$ vanish, and the geodesic surface $S$ is totally geodesic.

Further the curvature tensor $R_{\alpha\beta\gamma\delta}$ of $S$ is given by the Gauss equation; it is now of the simple form

$$R_{\alpha\beta\gamma\delta}=S_{hijk}(z)Z_\alpha{}^hZ_\beta{}^iZ_\gamma{}^jZ_\delta{}^k.$$

Thus $R_{1212}=0$ is easily seen by the homogeneity of $S_{hijk}(y)$ and the equation $S_{hijk}(y)y^k=0$, so that we have $R_{\alpha\beta\gamma\delta}=0$, and $S$ is flat.

To derive more interesting property of the geodesic surface $S$, we deal with the extrinsic equation of a geodesic on the indicatrix $I_x(L(y)=1)$ at the point $x$ of $F^n$. The theory of indicatrix as a hypersurface of the Riemannian space $M_x{}^n$ has been developed in *§31. Let $y=y(u^1,\cdots,u^{n-1})$ be the parametric equation of $I_x$ and put $Y_\alpha=\partial y/\partial u^\alpha$ $(\alpha=1,\cdots,n-1)$. Then the Gauss formula of $I_x$ (*(31.6$_1$)) is written as

$$\frac{\partial Y_\alpha}{\partial u^\beta}+C(Y_\alpha, Y_\beta)=\Gamma_\alpha{}^\gamma{}_\beta Y_\gamma-g_{\alpha\beta}y,$$

where $g_{\alpha\beta}$ and $\Gamma_\alpha{}^\gamma{}_\beta$ $(\alpha, \beta, \gamma=1, \cdots, n-1)$ are components of the induced fundamental tensor and the Christoffel symbols of $I_x$ respectively.

Let $\gamma=y(\sigma)$ be a curve on $I_x$ with the arc length $\sigma$ as the parameter. We multiply the above by $(du^\alpha/d\sigma)(du^\beta/d\sigma)$ to obtain

$$D_\sigma y_\sigma=-y+Y_\alpha\delta_\sigma{}^2u^\alpha,$$

where $D_\sigma$ is the extrinsic covariant differentiation along $\gamma$ in $M_x{}^n$ and $\delta_\sigma$ is the intrinsic one along $\gamma$ in $I_x$:

$$\delta_\sigma{}^2u^\alpha=\frac{d^2u^\alpha}{d\sigma^2}+\Gamma_\beta{}^\alpha{}_\gamma\frac{du^\beta}{d\sigma}\frac{du^\gamma}{d\sigma}.$$

The curve $\gamma$ is a geodesic on $I_x$, iff $\delta_\sigma{}^2u^\alpha=0$ holds good. Consequently we have

PROPOSITION 2. *In terms of the coordinate* $(y^i)$ *in* $M_x{}^n$ *a geodesic on the indicatrix is given by the equation* $D_\sigma y_\sigma=-y$, *where* $\sigma$ *is the arc length of the geodesic.*

We now return to the consideration of the geodesic surface $S$ generated by the normal geodesic $C=y(s)$. Let $\gamma=Y(\sigma)$ be the intersection of $S$ with $I_x$. Then $Y(\sigma)$ is given by the equation

$$Y(\sigma)=\frac{y(s)}{L(y(s))}. \tag{4.1}$$

Differentiation of (4.1) by the arc length $s$ of $C$ yields

$$L'Y+L\sigma'Y_\sigma=y_s. \tag{4.2}$$

It follows from (4.2) that the equation (1.6) is written as

$$(L')^2\langle Y, Y\rangle_y+2LL'\sigma'\langle Y, Y_\sigma\rangle_y+(L\sigma')^2\langle Y_\sigma, Y_\sigma\rangle_y=1.$$

Although the inner products in the above are evaluated at the point $y(s)$ of $C$, the homogeneity of $g_{ij}(y)$ and (4.1) yield $g_{ij}(y)=g_{ij}(Y)$, so that the above is written as

$$(\sigma')^2=\frac{1-(L')^2}{L^2}. \tag{4.3}$$

Next, differentiation of (4.2) yields

$$L''Y+(2L'\sigma'+L\sigma'')Y_\sigma+L(\sigma')^2D_\sigma Y_\sigma=D_sy_s. \tag{4.4}$$

The right-hand side of (4.4) vanishes by the assumption on $C$. Further the equation in Proposition 1 and (4.3) give

$$(\sigma')^2=\frac{h^2}{L^4},\quad \sigma'\sigma''=-\frac{2h^2(s-c)}{L^6}. \tag{4.5}$$

Hence (4.4) is reduced to $D_\sigma Y_\sigma+Y=0$, and the intersection $\gamma$ is a geodesic on $I_x$.

We conversely consider a surface $S$ generated by a geodesic $\gamma=Y(\sigma)$ on $I_x$. The equation of $S$ is given by $z=\tau Y(\sigma)$ with a positive-valued parameter $\tau$, and we deal with a curve $C=y(s)=\tau(\sigma)Y(\sigma)$ on $S$. The equation is obviously written in the form $y(s)=LY(\sigma)$, where $L=L(y(s))$. Then the equations (4.2), (4.3) and (4.4) are derived. Since $\gamma$ is a geodesic, (4.4) is written in the form

$$(2L'\sigma'+L\sigma'')Y_\sigma+[L''-L(\sigma')^2]Y=D_s y_s.$$

By (4.3) it is seen that the coefficients of the above are written as

$$2L'\sigma'+L\sigma''=-\frac{L'[(L^2)''-2]}{2L^2\sigma'},$$

$$L''-L(\sigma')^2=\frac{(L^2)''-2}{2L}.$$

As a consequence, if we choose the curve $C$ such that the length $L(y(s))$ satisfies $(L^2)''=2$, then $D_s y_s$ holds good, and $C$ is a geodesic in $M_x^n$. It is clear that $L^2=(s-c)^2+h^2$ satisfies $(L^2)''=2$.

Summarizing up the above results, we establish

THEOREM 5. *Every geodesic surface is totally geodesic and flat. Every geodesic on the indicatrix $I_x$ is the intersection of $I_x$ with a geodesic surface in $M_x^n$.*

This relation between geodesic surfaces in $M_x^n$ and geodesics on $I_x$ is similar to the following fact in a euclidean space: Every geodesic on the sphere is the intersection of the sphere with a plane through the centre.

## § 5. On Jacobi fields along rays

We are concerned with Jacobi fields along a ray $C=y(s)$ which is given by the equation (2.1). Then we have already seen $C_j{}^i{}_k(y)y_s{}^k=0$ in § 2. It follows from (1.3) and (1.4′) that a Jacobi field $J$ along $C$ is given solely by the differential equation $d^2J/ds^2=0$. Consequently

PROPOSITION 3. *The components $J^i$ of any Jacobi field $J$ along a ray $C$ are linear functions of the arc length $s$ of $C$.*

Therefore we first conclude immediately

LEMMA 3. *Any ray has no pair of conjugate points and is the relative shortest geodesic.*

Let $P_1(y_1=y(s_1))$ and $P_2(y_2=y(s_2))$ be two points of the ray $C$ with the equation (2.1). The arc length $\overline{P_1P_2}$ of $C$ from $P_1$ to $P_2$ is given by

$$\overline{P_1P_2}=|s_2-s_1|=|L(y_2)-L(y_1)|. \tag{5.1}$$

We indicate by $C^0$ the ray $C$ with the origin $O$ added. From the above point of view or by the improper integral of the arc length the length $\overline{OP_1}$ of $C^0$ from $O$ to $P_1$ may be defined as $\overline{OP_1}=L(y_1)$.

In the following we denote by $\overline{PQ}$ the arc length of a ray from a point $P$ to a point $Q$. On the other hand the notation $\widehat{PQ}$ stands for the arc length of arbitrary curve from $P$ to $Q$.

Let $\gamma$ be an abnormal curve and $\gamma^0$ the curve with the origin $O$ added. The arc length $\widehat{OP}$ of $\gamma^0$ is similarly defined by the improper integral. Or it is defined as follows: Let $\{P_i\}$ $(i=1,2,\cdots)$ be a Cauchy point sequence on $\gamma$ which convergers to $O$. Then $\widehat{OP}$ is equal to the limit of $\overline{OP_i}+\widehat{P_iP}$.

First we compare the arc length $\overline{OP}$ of the ray with the length $\widehat{OP}$ of a curve $\gamma^0$ from $O$ to $P$, where $\gamma$ is assumed to be enough near to $C$ so that the following inequalities hold good. Let $\{P_i\}$ and $\{Q_i\}$ be Cauchy point sequences on $C$ and $\gamma$ respectively which converge to $O$. We combine $P_i$ with $Q_i$ by a small arc $\gamma_i$. Then Lemma 3 leads us to the inequalities

$$\overline{P_iP}<\widehat{P_iQ_i}+\widehat{Q_iP}.$$

As the limit of the inequalities we obtain

$$\overline{OP}\leq\widehat{OP}. \tag{5.2}$$

Next, we compare $\overline{PQ}$ with the arc length $\widehat{PQ}$ of arbitrary curve $\gamma$ from $P$ to $Q$. Take a subdivision $P=P_0, P_1, \cdots, P_n=Q$ of $\gamma$, which is assumed to be enough fine so that the following inequalities hold good. By (5.2) we obtain

$$\overline{OP_i}\leq\overline{OP_{i+1}}+\widehat{P_{i+1}P_i}\ \ (i=0,1,\cdots,\ n-1).$$

We sum up all the above inequalities to obtain $\overline{OP}\leq\overline{OQ}+\widehat{PQ}$.

Thus (5.1) leads us to

$$\overline{PQ}\leq\widehat{PQ}. \tag{5.3}$$

Consequently any ray is the shortest geodesic. If the equality holds in (5.3), then $\gamma$ must be geodesic. In this case we take three points $P, Q, R$ on the ray in this order and denote by $\underline{\gamma}$ the $\gamma$ with the arc $QR$ of the ray added. Then $\overline{PR}=$(arc length of $\gamma$), so that $\gamma$ is also a shortest geodesic and must be smooth. Therefore $\gamma$ must coincides with the ray.

Consequently we can conclude

THEOREM 6. *Any ray is the unique shortest geodesic, combining any two points of it.*

## §6. On Jacobi fields along normal geodesics

The following properties of Jacobi fields along geodesics are well-known in the general theory of geodesics in Riemannian spaces:

LEMMA 4. (1) *The vertical component of a Jacobi field is also a Jacobi field.* (2) *The tangent vector field $vy_s$ of a geodesic $y(s)$ is a Jacobi field, iff $v$ is a linear function of $s$.* (3) *If $J$ is a Jacobi field along a geodesic $y(s)$, then $\langle J, y_s\rangle$ is a linear function of $s$.* (4) *If a Jacobi field $J$ along a geodesic $C$ vanishes at two points of $C$, then $J$ is orthogonal to $C$ at each point of $C$.*

These are easily shown, but we shall see the proofs in order to know the reasons. First the vertical component $V$ of a Jacobi field $J$ along a geodesic $C=y(s)$ is given by

$$V=J-\langle J, y_s\rangle y_s. \tag{6.1}$$

This is really vertical, that is,

$$\langle V, y_s\rangle=0. \tag{6.2}$$

It follows from (1.1′) and (1.4) that

$$\begin{aligned} D_s^2 V&=D_s^2 J-\langle D_s^2 J, y_s\rangle y_s\\ &=-S(y_s, y_s, V+\langle J, y_s\rangle y_s)+\langle S(y_s, y_s, J), y_s\rangle y_s\\ &=-S(y_s, y_s, V). \end{aligned}$$

Thus (1) is proved. In the above process only the usual skew-symmetry property $S_{hijk}=-S_{hikj}=-S_{ihjk}$ is effective. Secondly we have $D_s^2(vy_s)=v''y_s$ and $S(y_s, y_s, vy_s)=0$ by $S_{hijk}=-S_{hikj}$, hence (2) is proved. Thirdly it is seen that $(\langle J, y_s\rangle)''=\langle D_s^2 J, y_s\rangle=-\langle S(y_s, y_s, J), y_s\rangle=0$ by $S_{hijk}=-S_{ihjk}$, hence (3) is proved. Finally, if $J$ vanishes at two points of $C$, then $\langle J, y_s\rangle$ vanishes at these points, hence (3) proves (4).

Consequently we notice that special characters of the $v$-curvature tensor $S$ of the Finsler space $F^n$ do not play a role in the proofs of Lemma 4. In the following, however, those special characters are essentially effective.

PROPOSITION 4. *A vector field $uy(s)$ is a Jacobi field along a normal geodesic $C=y(s)$ in $M_x^n$, iff $u$ is a constant.* (Such a Jacobi field is said to be *radial.*)

In fact, it is seen for $J=uy(s)$ that $D_s^2J=u''y+2u'y_s$ and $S(y_s, y_s, J)=0$ by the special property of $S$, i.e., $S_h{}^i{}_{jk}(y)y^k=0$. Hence (1.4) is written as $u''y+2u'y_s=0$. Thus Theorem 1 completes the proof.

The existence of radial Jacobi field leads us to the concept of *parallel normal geodesics*. That is, if $C=y(s)$ is a normal geodesic, then the curve $C'$ with the radius vector $uy(s)$ for a positive constant $u$ is also a normal geodesic. $C'$ may be called parallel to $C$.

Let us consider a general Jacobi field $J$ along a normal geodesic $C=y(s)$. Because the vertical component $V$ of $J$ is a Jacobi field, we have $D_s^2V+S(y_s, y_s, V)=0$, which implies

$$\langle D_s^2 V, y\rangle=0, \tag{6.3}$$

because of the equation $S_{hijk}(y)y^i=0$.

From (6.2) and (6.3) we derive $\langle D_sV, y_s\rangle=0$. Therefore $(\langle V, y\rangle)''=0$ is easily verified, and we have

$$\langle V, y\rangle=a_0s+a_1, \tag{6.4}$$

where $a_0$ and $a_1$ are constants.

We can put $\langle J, y_s\rangle=b_0s+b_1$ by (3) of Lemma 4, where $b_0$ and $b_1$ are constants. Consequently (6.1) and (6.4) together with Proposition 1 yield

$$\langle J, y\rangle=a_0s+a_1+(b_0s+b_1)(s-c). \tag{6.5}$$

PROPOSITION 5. *Any Jacobi field $J$ along a normal geodesic $C=y(s)$ in $M_x^n$ satisfies the equation* (6.5), *where*

$$\langle V, y\rangle=a_0s+a_1, \quad \langle J, y_s\rangle=b_0s+b_1,$$

*$V$ is the vertical component of $J$ and $a_0, a_1, b_0, b_1$ are constants.*

We are concerned with a conjugate point $P_2(y_2=y(s_2))$ to a point $P_1(y_1=y(s_1))$ along a normal geodesic $C=y(s)$. Then there exists a non-zero Jacobi field $J$ along $C$, which vanishes at $P_1$ and $P_2$. (4) of Lemma 4 shows

$$\langle J, y_s\rangle=0, \tag{6.6}$$

which implies $b_0=b_1=0$ and further $a_0=a_1=0$ in Proposition 5, because $J=V$. Consequently $J$ is orthogonal to $y$ as well as $y_s$.

LEMMA 5. *If a vector field $U$ along a normal geodesic $C=y(s)$ is orthogonal to the geodesic surface $S$ generated by $C$, then $D_sU$ is also orthogonal to $S$.*

In fact, the assumption means $\langle U, y\rangle=\langle U, y_s\rangle=0$. From these equations we immediately obtain $\langle D_sU, y\rangle=\langle D_sU, y_s\rangle=0$.

THEOREM 7. *In case of dimension more than two, if a point $P_2$ is a conjugate point to a point $P_1$ along a normal geodesic $C$, then the non-zero Jacobi field $J$ along $C$, which vanishes at $P_1$ and $P_2$ as well as $D_sJ$, are orthogonal to the geodesic surface generated by $C$.*

REMARK. Although Theorem 4 has been shown in virtue of the general theorem of Schoenberg, the above result also proves Theorem 4, because in $M_x^2$ a vector orthogonal to linearly independent vectors $y(s)$ and $y_s$ must be equal to zero.

## §7. Jacobi fields along geodesics on the indicatrix

In viewpoint of Theorem 7 we are interested in a Jacobi field along a normal geodesic $C$, which is orthogonal to the geodesic surface generated by $C$. To generalize (2) of Lemma 4 and Proposition 4, we first show

LEMMA 6. *The vector field $uy+vy_s$ is a Jacobi field along a normal geodesic $C=y(s)$, iff $u(s)$ and $v(s)$ satisfy the differential equations $u''=0$ and $v''+2u'=0$.*

In fact, from $J=uy+vy_s$ we obtain

$$D_s^2J=u''y+(2u'+v'')y_s, \quad S(y_s, y_s, J)=0.$$

Hence Lemma 1 completes the proof.

We now put

$$U=J-uy-vy_s \tag{7.1}$$

for any Jacobi field $J$ along a normal geodesic $C=y(s)$. It follows from Proposition 1 that $U$ is orthogonal to $y$ and $y_s$, iff

$$\langle U, y\rangle=\langle J, y\rangle-uL^2-v(s-c)=0,$$

$$\langle U, y_s\rangle=\langle J, y_s\rangle-u(s-c)-v=0.$$

Hence we obtain by Propositions 1 and 5

$$\begin{aligned} h^2u&=a_0s+a_1, \\ h^2v&=h^2(b_0s+b_1)-(a_0s+a_1)(s-c). \end{aligned} \tag{7.2}$$

It is easy to verify that these $u$ and $v$ satisfy the condition in Lemma 6, hence $U$ given by (7.1) is a Jacobi field. Therefore

PROPOSITION 6. *Let $J$ be a Jacobi field along a normal geodesic $C=y(s)$. Then $U$ defined by (7.1) with (7.2) is a Jacobi field along $C$ and is orthogonal to the geodesic surface generated by $C$, where the notations of Propositions*

1 *and* 5 *are used.*

In §4 we treated a geodesic $\gamma$ on the indicatrix $I_x$ in terms of the coordinate $(y^i)$ of $M_x^n$. To consider Jacobi fields along $\gamma$, we deal with a vector field $V$ in $M_x^n$ which is defined along a general curve $\gamma=u(\sigma)$ on $I_x$. Suppose that it is written as

$$V=v^\alpha Y_\alpha+v^0 y,$$

where $v^\alpha$ are the tangential components to $I_x$ and $v^0$ is the normal component to $I_x$. In the notations of §4 and by means of *(31.6) we obtain the formula of the covariant derivative $D_\sigma V$ along $\gamma$:

$$\text{(7.3)} \qquad D_\sigma V=(\delta_\sigma v^\alpha+v^0\frac{du^\alpha}{d\sigma})Y_\alpha+(\frac{dv^0}{d\sigma}-\langle v,\ \frac{du}{d\sigma}\rangle)y,$$

where $\langle v,\ du/d\sigma\rangle$ is the inner product on $I_x$.

Now we are concerned with a Jacobi field $K^i=\kappa^\alpha Y_\alpha^i$ along a geodesic $\gamma=u(\sigma)$ with the extrinsic equation $y(\sigma)$. By (7.3) we first obtain

$$\text{(7.4)} \qquad D_\sigma K=\delta_\sigma\kappa^\alpha Y_\alpha-\langle\kappa,\ \frac{du}{d\sigma}\rangle y.$$

By (7.3) again we obtain from (7.4)

$$\text{(7.5)} \qquad D_\sigma^2 K=(\delta_\sigma^2\kappa^\alpha-\langle\kappa,\ \frac{du}{d\sigma}\rangle\frac{du^\alpha}{d\sigma})Y_\alpha-2\langle d\sigma\kappa,\ \frac{du}{d\sigma}\rangle y.$$

As $\kappa$ is a Jacobi field, it satisfies the differential equations similar to (1.4):

$$\delta_\sigma^2\kappa^\alpha+R_\beta{}^\alpha{}_{\gamma\delta}\frac{du^\beta}{d\sigma}\frac{du^\gamma}{d\sigma}\kappa^\delta=0,$$

where $R_\beta{}^\alpha{}_{\gamma\delta}$ are components of the curvature tensor of $I_x$; they are given by the Gauss equation *(31.7). Therefore we have

$$R_\beta{}^\alpha{}_{\gamma\delta}\frac{du^\beta}{d\sigma}\frac{du^\gamma}{d\sigma}\kappa^\delta Y_\alpha^i=S_h{}^i{}_{jk}Y_\sigma^h Y_\sigma^j K^k+K^i-\langle\kappa,\ \frac{du}{d\sigma}\rangle\ Y_\sigma^i.$$

On the other hand we have $\langle D_\sigma K, Y_\sigma\rangle=\langle\delta_\sigma\kappa,\ du/d\sigma\rangle$ from (7.4). Consequently (7.5) is written in the form

$$\text{(7.5}'\text{)} \qquad D_\sigma^2 K+K+S(Y_\sigma,\ Y_\sigma,\ K)+2\langle D_\sigma K,\ Y_\sigma\rangle y=0.$$

This is the extrinsic equation of a Jacobi field $K$ along a geodesic $y(\sigma)$ on $I_x$.

We now consider a Jacobi field $J$ along a normal geodesic $C=y(s)$, which is orthogonal to the radius vector $y(s)$. By Theorem 5 the intersection $\gamma$ of $I_x$ with the geodesic surface $S$ generated by $C$ is a geodesic on $I_x$. The geodesic $\gamma$ is given by the equation (4.1), and there is the relation (4.5) between the arc length $s$ of $C$ and $\sigma$ of $\gamma$. Since $J$ is orthogonal to $y(s)$, it is regarded as a tangent vector to $I_x$ at the point $Y(\sigma)$. We shall find a condition for $K=k(s)J(s)$ to be a Jacobi field along $\gamma$. We see

$$\sigma' D_\sigma K = k'J + kD_sJ,$$

$$\sigma'' D_\sigma K + (\sigma')^2 D_\sigma^2 K = k''J + 2k'D_sJ + kD_s^2J.$$

Paying attention to (1.4), we obtain the equation (7.6) from the above. In this process it is remarked that the tensor $S$ in (1.4) is evaluated at the point $y(s)$, i.e., $S_y(y_s, y_s, J)$. Therefore

$$S_y(y_s, y_s, J) = S_{LY}(L'Y + L\sigma' Y_\sigma, L'Y + L\sigma' Y_\sigma, K/k);$$

this is equal to $(\sigma')^2 S_Y(Y_\sigma, Y_\sigma, K)/k$. Consequently we obtain

$$(7.6)\qquad \frac{h^2}{L^4}[D_\sigma^2 K + S(Y_\sigma, Y_\sigma, K)] + \left[2\left(\frac{k'}{k}\right)^2 - \frac{k''}{k}\right] K$$

$$-2\sigma'\left(\frac{s-c}{L^2} + \frac{k'}{k}\right) D_\sigma K = 0.$$

Comparing (7.6) with (7.5′), it is finally seen that $K=kJ$ is a Jacobi field along $\gamma$, iff the following equations hold good:

$$(7.7)\qquad \begin{aligned} &(1)\quad 2\left(\frac{k'}{k}\right)^2 - \frac{k''}{k} = \frac{h^2}{L^4},\\ &(2)\quad \frac{s-c}{L^2} + \frac{k'}{k} = 0,\\ &(3)\quad \langle D_\sigma K, Y_\sigma\rangle = 0. \end{aligned}$$

First from (2) we obtain $k=k_0/L$ with a constant $k_0$, and (1) is automatically satisfied. Then (3) is equivalent to $\langle D_sJ, y_s\rangle=0$ by the equation $\langle D_s J, y\rangle + \langle J, y_s\rangle = 0$, which is derived from the assumption $\langle J, y\rangle=0$.

The converse will be easily shown. We summarize up the above result as follows:

THEOREM 8. *Let $\gamma$ be the intersection of the indicatrix with the geodesic surface generated by a normal geodesic $C=y(s)$ in $M_x^n$ $(n>2)$. Given a non-zero constant $k_0$, there is the one-to-one correspondence between a Jacobi field $J$ along $C$ satisfying $\langle J, y(s)\rangle = \langle D_sJ, y_s\rangle = 0$ and a Jacobi field $\kappa$ along $\gamma$ satisfying $\langle \delta_\sigma \kappa, du/d\sigma\rangle = 0$; the correspondence is given by $k_0 J^i = L(y(s))\kappa^\alpha Y_\alpha^i$.*

Suppose that a point $y_2$ is a conjugate point to $y_1$ along the normal geodesic $C=y(s)$. Theorem 7 shows the existence of a Jacobi field $J$ which satisfies the condition stated in Theorem 8 and vanishes at $y_1$ and $y_2$. Therefore it is seen from Theorem 8 that the corresponding points $Y_1$ and $Y_2$ of the geodesic $\gamma$ are conjugate. Suppose conversely that $Y_2$ is a conjugate point to a point $Y_1$ along $\gamma$. Then there exists a Jacobi field $\kappa$ along $\gamma$ which vanishes at $Y_1$ and $Y_2$, and (4) of Lemma 4 shows that $\kappa$ satisfies the condition in Theorem 8, because of $\langle\kappa, du/d\sigma\rangle=0$. Therefore Theorem 8 shows that the points $y_1$ and $y_2$ of $C$ corresponding to $Y_1$ and $Y_2$ respec tively are conjugate along $C$.

The above fact holds good for any normal geodesic, by which the geodesic surface under consideration is generated.

Consequently we can state

COROLLARY. *Let $S$ be the geodesic surface generated by a geodesic $\gamma$ on the indicatrix, and let $Y_2$ be a conjugate point to a point $Y_1$ along $\gamma$. Then the point of intersection of any normal geodesic $C$ on $S$ with the ray through $Y_2$ is conjugate to the point of intersection of $C$ with the ray through $Y_1$.*

Thus we may imagine such a figure as a folding fan; the pivot is the origin of $F_x^n$, the paper is a geodesic surface and the ribs are rays consisting of conjugate points along any normal geodesic on the paper. It is well-known (*Theorem 41.2) that a conjugate point $P$ to $Q$ is, roughly speaking, such that there exists a field of geodesics with one-parameter which go from $P$ to $Q$. Therefore many fans with common ribs are in layers.

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