J. Korean Math. Soc.
Vol. 14, No. 2, 1978.

ON INITIALLY STRUCTURED FUNCTORS

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1. Introduction.

It is well known that initial and final structures in the sense of Bourbaki [1] play the crucial role in the study of various mathematical structures. Herrlich has introduced topological functors [4, 6] for the study of topological structures by means of categorical methods.

We have introduced topologically algebraic functors [9, 10] for the study of topological algebras and shown that most of underlying set functors from categories of topological algebras are topologically algebraic. In this paper we pursue further properties of topologically algebraic functors, in particular factorization properties of those functors.

In the first section, we investigate the relationship of initial sources for the case of factorized functors. In the second section, we show that algebraic categories are topologically algebraic and that the compositions of topological functors with topologically algebraic functors are also topologically algebraic. Hence the compositions of topological functors with algebraic functors into **Set**, $(\boldsymbol{E}, \boldsymbol{M})$ topological functors, regular functors and monadic functors into **Set** are all topologically algebraic, so that topologically algebraic functors are in a way compositions of underlying topological space functors and underlying algebra functors. Finally it is shown that for any topologically algebraic functor and the monad $\boldsymbol{T}$ generated by its adjoint situation, the comparison functor to the category of $\boldsymbol{T}$-algebras is also topologically algebraic.

2. Initial sources.

The following definition is due to Herrlich [6].

2.1 DEFINITION. Let $U:\boldsymbol{A}\to\boldsymbol{B}$ be a functor. A source $(A, (f_i:A\to A_i)_{i\in I})$ in $\boldsymbol{A}$ is called *U-initial* (or *initial with respect to* U) if for any source $(A', (g_i:A'\to A_i)_{i\in I})$ in $\boldsymbol{A}$ and any $\boldsymbol{B}$-morphism $u:UA'\to UA$ with $(Uf_i)u=Ug_i$ $(i\in I)$ there exists a unique $\boldsymbol{A}$-morphism $g:A'\to A$ with $Ug=u$ and $f_ig=g_i$

Received by the editors June 1, 1977.
Financial assistance from Korean Traders Scholarship Foundation Grant 1976 is gratefully acknowleged.

$(i\in I)$.

2.2 REMARK. 1) For the category **Top** (**Unif**) of topological (uniform, resp.) spaces and (uniformly, resp.) continuous maps and the underlying set functor U:**Top**$\to$**Set** (**Unif**$\to$**Set**, resp.), a source $(A, (f_i)_I)$ in **Top** (**Unif**, resp.) is U-initial iff A is endowed with the initial topology (uniform structure, resp.) with respect to $(f_i)_I$ (see [1, 6]).

2) For a category $\boldsymbol{A}$ of universal algebras of fixed type and homomorphisms and the underlying set functor $U:\boldsymbol{A}\to$**Set**, every mono-source $(A, (f_i)_I)$ in $\boldsymbol{A}$ is U-initial (see [9]).

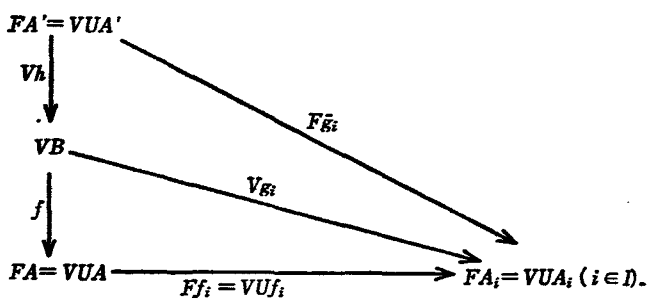
2.3 PROPOSITION. *Let $U:\boldsymbol{A}\to\boldsymbol{B}$ and $V:\boldsymbol{B}\to\boldsymbol{C}$ be functors and $F=VU$. If V is faithful, then every F-initial source $(A, (f_i:A\to A_i)_{i\in I})$ in $\boldsymbol{A}$ is also U-initial.*

Proof. Take a source $(A', (g_i:A'\to A_i)_{i\in I})$ and a $\boldsymbol{B}$-morphism $f:UA'\to UA$ such that $(Uf_i)f=Ug_i$ $(i\in I)$. Since a functor preserves commuting diagrams, $(VUf_i)(Vf)=VUg_i$, i.e. $(Ff_i)(Vf)=Fg_i$ $(i\in I)$. Since $(A, (f_i)_I)$ is F-initial, there is a unique morphism $g:A'\to A$ in $\boldsymbol{A}$ with $Fg=VUg=Vf$ and $f_ig=g_i$ $(i\in I)$; therefore $Ug=f$ and $f_ig=g_i$ $(i\in I)$ for V is faithful. Suppose $g':A'\to A$ is a morphism in $\boldsymbol{A}$ with $Ug'=f$ and $f_ig'=g_i$ $(i\in I)$. Then $Fg'=V(Ug')=Vf$ and $f_ig'=g_i$ $(i\in I)$; hence $g'=g$.

2.4 PROPOSITION. *Let $U:\boldsymbol{A}\to\boldsymbol{B}$ and $V:\boldsymbol{B}\to\boldsymbol{C}$ be functors and $F=VU$. Let $(A, (f_i:A\to A_i)_{i\in I})$ be a source in $\boldsymbol{A}$. Suppose U is dense and full, and V is faithful or $(UA, (Uf_i)_I)$ is a monosource. Then $(UA, (Uf_i)_I)$ is V-initial if $(A, (f_i)_I)$ is F-initial.*

Proof. Let $(B, (g_i:B\to UA_i)_{i\in I})$ be a source in $\boldsymbol{B}$ and $f:VB\to VUA$ a $\boldsymbol{C}$-morphism with $V(Uf_i)f=Vg_i$ $(i\in I)$. Since U is dense, there is an $\boldsymbol{A}$-object A' and an isomorphism $h:UA'\to B$. Since U is full and $g_ih:UA'\to UA_i$, there is an $\boldsymbol{A}$-morphism $\bar{g}_i:A'\to A_i$ with $U\bar{g}_i=g_ih$ $(i\in I)$.

Then we have the following commuting diagram



Since $(A, (f_i)_I)$ is F-initial, there is a unique $\boldsymbol{A}$-morphism $g': A' \to A$ with $F(g') = VU(g') = fV(h)$ and $f_i g' = \bar{g}_i$ $(i \in I)$. Let $g = (Ug')h^{-1}: B \to UA$, then $Vg = (VUg')Vh^{-1} = (Fg')Vh^{-1} = f$ and $(Uf_i)g = g_i$ $(i \in I)$. Using the fact that V is faithful or $(UA, (Uf_i)_I)$ is a monosource, it is obvious that g is in fact unique.

2.5 Proposition. *Let $U: \boldsymbol{A} \to \boldsymbol{B}$ and $V: \boldsymbol{B} \to \boldsymbol{C}$ be functors and $F = VU$. Suppose U is full and faithful and $(A, (f_i: A \to A_i)_{i \in I})$ is a source in $\boldsymbol{A}$. If $(UA, (Uf_i)_I)$ is V-initial then $(A, (f_i)_I)$ is also F-initial.*

Proof. For any source $(A', (g_i: A' \to A_i)_{i \in I})$ and a $\boldsymbol{C}$-morphism $f: FA' \to FA$ with $(Ff_i)f = Fg_i$ $(i \in I)$, there is a unique morphism $g': UA' \to UA$ such that $Vg' = f$ and $g'(Uf_i) = Ug_i$ $(i \in I)$, for $(UA, (Uf_i)_I)$ is V-initial and $(V(Uf_i))f = V(Ug_i)$ $(i \in I)$. Since U is full, there is an $\boldsymbol{A}$-morphism $g: A' \to A$ with $Ug = g'$; hence $Fg = f$ and $U(gf_i) = Ug_i$ $(i \in I)$. Using the fact that U is faithful, $gf_i = g_i$ and g is unique.

2.6 Proposition. *Let $U: \boldsymbol{A} \to \boldsymbol{B}$ and $V: \boldsymbol{B} \to \boldsymbol{C}$ be functors and $F = VU$. Suppose a source $(A, (f_i: A \to A_i)_{i \in I})$ in $\boldsymbol{A}$ is U-initial and $(UA, (Uf_i)_I)$ is V-initial. If F is faithful or $(A, (f_i)_I)$ is a monosource, then $(A, (f_i)_I)$ is also F-initial.*

Proof. Take a source $(A', (g_i: A' \to A_i)_{i \in I})$ in $\boldsymbol{A}$ and a $\boldsymbol{C}$-morphism g: $FA' \to FA$ such that $Fg_i = (Ff_i)g$ $(i \in I)$. Since $V(Uf_i)g = V(Ug_i)$, there is a unique $\boldsymbol{B}$-morphism $g': UA' \to UA$ with $Vg' = g$ and $(Uf_i)g' = Ug_i$ $(i \in I)$; therefore there is again a unique $\boldsymbol{A}$-moprhism $\bar{g}: A' \to A$ such that $U\bar{g} = g'$ and $f_i \bar{g} = g_i$ $(i \in I)$. It is then obvious that $F\bar{g} = VU\bar{g} = Vg' = g$ and $f_i \bar{g} = g_i$ $(i \in I)$, and $\bar{g}$ is unique, for F is faithful or $(A, (f_i)_I)$ is a monosource in $\boldsymbol{A}$.

3. Topologically algebraic functors.

3.1 Definition. ([9, 10]) A functor $U: \boldsymbol{A} \to \boldsymbol{B}$ is called *topologically algebraic* if for each family $(A_i)_{i \in I}$ of $\boldsymbol{A}$-objects and each source $(B, (g_i: B \to UA_i)_{i \in I})$ in $\boldsymbol{B}$, there exists a U-initial source $(A, (f_i: A \to A_i)_{i \in I})$ in $\boldsymbol{A}$ and a U-generating morphism $h: B \to UA$ such that $(Uf_i)h = g_i$ for all $i \in I$.

The following definition is due to Herrlich [6].

3.2 Definition. Let $\boldsymbol{B}$ be an $(\boldsymbol{E}, \boldsymbol{M})$ category and $U: \boldsymbol{A} \to \boldsymbol{B}$ a functor. U is called $(\boldsymbol{E}, \boldsymbol{M})$ topological if for each family $(A_i)_{i \in I}$ of $\boldsymbol{A}$-objects and each source $(B, (m_i: B \to UA_i)_{i \in I})$ in $\boldsymbol{M}$ there is a U-initial source $(A, (f_i: A \to A_i)_{i \in I})$ and an isomorphism $h: B \to UA$ with $(Uf_i)h = m_i$ for each $i \in I$. In particular, for the class $\boldsymbol{E}$ of all isomorphisms and the class $\boldsymbol{M}$ of all sources in $\boldsymbol{B}$, $(\boldsymbol{E}, \boldsymbol{M})$ topological functor is called topological.

It is known that the underlying set functor U:**Top**→**Set** (**Unif**→**Set**) is topological and that for the category **Haus** (**HUnif**) of Hausdorff (uniform, resp.) spaces and (uniformly, resp.) continuous maps, its underlying set functor is (epi, monosources) topological. Hence the concept of $(\boldsymbol{E}, \boldsymbol{M})$ topological functors is the most pertinent concept for the study of topology. For any category $\boldsymbol{A}$ of universal algebras of fixed type and homomorphisms which is closed under products and subalgebras, and for any epi-reflective subcategory $\boldsymbol{T}$ of **Top** (**Unif**), the category of topological (uniform, resp.) algebras whose underlying algebras belong to $\boldsymbol{A}$ and underlying spaces belong to $\boldsymbol{T}$ will be denoted by $\boldsymbol{TA}$. Then it is also known [9] that the underlying set functor U: $\boldsymbol{TA}$→**Set** is topologically algebraic. Therefore the concept of topologically algebraic functors is the counterpart of topological functors for the study of topological algebras. It is also known [9] that every $(\boldsymbol{E}, \boldsymbol{M})$ topological functor is topologically algebraic. The following definition is due to Herrlich [7].

3.3 Definition. A concrete category $(\boldsymbol{A}, U)$, i.e. U is a faithful functor from $\boldsymbol{A}$ to **Set**, is called *algebraic* if it satisfies the following three conditions:

1) $\boldsymbol{A}$ has coequalizers.
2) U has a left adjoint.
3) U preserves and reflects regular epimorphisms.

It is well known [7] that every monosource in an algebraic category $(\boldsymbol{A}, U)$ is U-initial, every algebraic category is an (extremal epi, monosource) category, and a full subcategory of the category of all universal algebras of fixed type and homomorphisms is algebraic iff it is epireflective in the category.

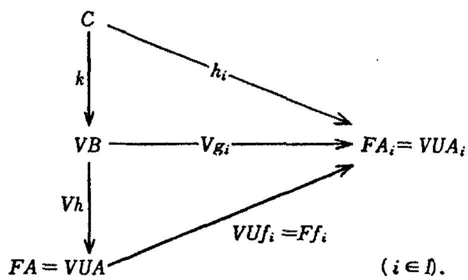
3.4 Theorem. *If* $(\boldsymbol{A}, U)$ *is an algebraic category, then* U:$\boldsymbol{A}$→**Set** *is a topologically algebraic functor.*

Proof. Let $(A_i)_{i\in I}$ be a family of $\boldsymbol{A}$-objects and $(X, (s_i: X\to UA_i)_{i\in I})$ a source in **Set**. Let F be a left adjoint of U and $\eta_X: X\to UFX$ the front adjunction for X. Then there exists a unique $\boldsymbol{A}$-morphism $\bar{s}_i: FX\to A_i$ such that $(U\bar{s}_i)\eta_X = s_i$ for each $i\in I$. Let $FX\xrightarrow{\bar{s}_i}A_i = FX\xrightarrow{e}A\xrightarrow{m_i}A_i$ be the (extremal epi, monosource)-factorization. Since $(A, (m_i)_I)$ is a U-initial source, and $(Ue)\eta_X$ is a U-generating morphism, U is topologically algebraic, for $(Um_i)((Ue)\eta_X) = s_i$ $(i\in I)$.

3.5 Theorem. *If a functor* U:$\boldsymbol{A}$→$\boldsymbol{B}$ *is topological and a functor* V:$\boldsymbol{B}$→$\boldsymbol{C}$

is topologically algebraic, then the functor $F=VU:\boldsymbol{A}\to\boldsymbol{C}$ is also topologically algebraic.

Proof. Let $(A_i)_{i\in I}$ be a family of $\boldsymbol{A}$-objects and $(C, (h_i:C\to FA_i)_{i\in I})$ a source in $\boldsymbol{C}$. For the source $(C, (h_i:C\to V(UA_i))_{i\in I})$, there exists a V-initial source $(B, (g_i:B\to UA_i)_{i\in I})$ and a V-generating morphism $k:C\to VB$ such that $(Vg_i)k=h_i$ for all $i\in I$. Again for the source $(B, (g_i)_I)$, there is a U-initial source $(A, (f_i:A\to A_i)_{i\in I})$ and an isomorphim $h:B\to UA$ with $(Uf_i)h=g_i$ $(i\in I)$. Hence we have the following commuting diagram



Since $Uf_i=g_ih^{-1}$ $(i\in I)$, $(UA, (Uf_i)_I)$ is V-initial. By the Proposition 2.6, $(A, (f_i)_I)$ is also F-initial. Moreover, $(Vh)k$ is F-generating. Indeed, for any pair (u,v) of $\boldsymbol{A}$-morphisms with $(Fu)(Vh)k=(Fv)(Vh)k$, $V(Uu)Vh=V(Uv)Vh$; $(Uu)h=(Uv)h$, for V is faithful (see [9]), and hence $u=v$. This completes the proof.

S.S. Hong has shown [8] that a regular functor (see [5]) and a monadic functor into **Set** are both topologically algebraic. Hence the following is immediate from the above Theorem.

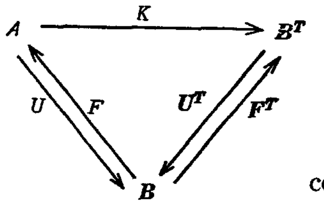
3.6 COROLLARY. *Let $U:\boldsymbol{A}\to\boldsymbol{B}$ be a topological functor. Suppose a functor $V:\boldsymbol{B}\to\boldsymbol{C}$ satisfies one of the following:*

1) *V is $(\boldsymbol{E},\boldsymbol{M})$ topological.*
2) *V is algebraic and $\boldsymbol{C}=$* **Set**.
3) *V is regular.*
4) *V is monadic and $\boldsymbol{C}=$* **Set**.

Then the composition functor $VU:\boldsymbol{A}\to\boldsymbol{C}$ is also topologically algebraic.

Let $(\eta,\varepsilon):F\dashv U:(\boldsymbol{A},\boldsymbol{B})$ be an adjoint situation and $\boldsymbol{T}=(UF,\mu,\eta)$ the monad generated by the adjoint situation (see [2]). We denote the category of $\boldsymbol{T}$-algebras by $\boldsymbol{B}^{\boldsymbol{T}}$ and the comparison functor by $K:\boldsymbol{A}\to\boldsymbol{B}^{\boldsymbol{T}}$. We recall that the objects of $\boldsymbol{B}^{\boldsymbol{T}}$ are (B,ξ), where B is a $\boldsymbol{B}$-object and $\xi:UFB\to B$ is a $\boldsymbol{B}$-morphism such that $\xi\eta_B=1_B$ and $\xi(UF\xi)=\xi\mu_B$, that the morphisms of $\boldsymbol{B}^{\boldsymbol{T}}$ are $f:(B,\xi)\to(B',\xi')$, where $f:B\to B'$ is a $\boldsymbol{B}$-morphism such that

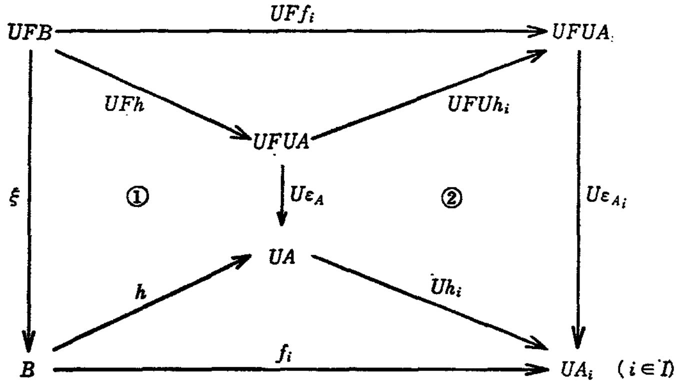
$\xi'(UFf)=f\xi$, and that $KA=(UA, U\varepsilon_A)$ and $Kf=Uf$. It is also well known [2] that



commutes, i.e. $U^TK=U$ and $FK=F^T$, and F^T is the left adjoint of U^T, and that U^T is monadic.

3.7 THEOREM. *If $U:\boldsymbol{A}\to\boldsymbol{B}$ is topologically algebraic and F is the left adjoint of U, then K is also topologically algebraic.*

Proof. Let $(A_i)_{i\in I}$ be a family of $\boldsymbol{A}$-objects and $((B,\xi), (f_i:(B,\xi)\to KA_i = (UA_i, U\varepsilon_{A_i}))_{i\in I})$ a source in $\boldsymbol{B}^T$. Then for the source $(B, (f_i:B\to UA_i)_{i\in I})$ in $\boldsymbol{B}$, there is a U-initial source $(A, (h_i:A\to A_i)_{i\in I})$ and a U-generating morphism $h:B\to UA$ such that $(Uh_i)h=f_i$ $(i\in I)$. Hence we have the following commuting diagram



Indeed, the outer rectangle is commuting, for $((B,\xi), (f_i)_I)$ is a source in $\boldsymbol{B}^T$. The diagram ② is commuting, for $\varepsilon:FU\to 1_{\boldsymbol{A}}$ is a natural transformation. Since $(Uh_i)h\xi=f_i\xi=U\varepsilon_{A_i}UFf_i=U\varepsilon_{A_i}UFUh_iUFh=(Uh_i)U(\varepsilon_AFh)=U(h_i\varepsilon_AFh)$ $(i\in I)$ and $(A, (h_i)_I)$ is U-initial, there is a unique $\boldsymbol{A}$-morphism $\bar{h}:FB\to A$ such that $U\bar{h}=h\xi$ and $h_i\bar{h}=h_i\varepsilon_AFh$. Moreover, $h=h\xi\eta_B$ and $U(\varepsilon_A Fh)\eta_B=U\varepsilon_AUFh\eta_B=U\varepsilon_A\eta_{UA}h=h$; therefore, $(U\bar{h})\eta_B=U(\varepsilon_AFh)\eta_B$. Hence $\bar{h}=\varepsilon_AFh$, so that $h\xi=U\varepsilon_AUFh$, i.e. the diagram ① is commuting. In all, $h:(B,\xi)\to(UA, U\varepsilon_A)$ is a morphism in $\boldsymbol{B}^T$ and h is also K-generating. Since $U=U^TK$ and U^T is faithful, $(A, (h_i)_I)$ is K-initial by Proposition 2.3. This completes the proof.

We have shown [9, 10] that every topologically algebraic functor has a left adjoint and detects limits and colimits. Hence the following is immediate from the above theorem.

3.8 COROLLARY. *If $U:\boldsymbol{A}\rightarrow\boldsymbol{B}$ is topologically algebraic and F is the left adjoint of U, then the comparison functor K has a left adjoint and detects limits and colimits.*

References

[1] N. Bourbaki, *Théorie des ensembles*, Chap. 4. Structures, Hermann, Paris (1957)
[2] S. Eilenberg and J.C. Moore, *Adjoint functors and triples*, Illinois J. Math. **9** (1965), 381-398
[3] T.H. Fay, *An axiomatic approach to categories of topological algebras*, Quaestiones Mathematicae **2** (1977), 133-137
[4] H. Herrlich, *Topological structures*, Math. Centre Tract, **52** (1974), 59-122
[5] H. Herrlich, *Regular categories and regular functors*, Canad. J. Math. **26** (1974), 709-720
[6] ______, *Topolgical functors*, Gen. Topology Appl. **4** (1974), 125-142
[7] ______ and G.E. Strecker, *Category Theory*, Allyn and Bacon, Boston (1973)
[8] S.S. Hong, *Categories in which every monosource is initial*, Kyungpook Math. J. **15** (1975), 133-139
[9] Y.H. Hong, *Studies on categories of universal topological algebras*, Doct. Dissertation, McMaster Univ. (1974)
[10] ______, *Topologically algebraic functors*, (preprint)

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