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AN INFINITESIMAL DEFORMATION CARRYING A HOLOMORPHICALLY PLANAR CURVE INTO A CURVE OF THE SAME KIND IN A KAEHLERIAN MANIFOLD

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§1. Introduction.

In a Riemannian manifold M with local coordinates $\{x^i\}$, we consider the point transformation

$$\bar{x}^i = x^i + \varepsilon v^i, \tag{1.1}$$

where ε is an infinitesimal constant and v^i is a vector field of M.

If the infinitesimal point transformation (1.1) under the condition

$$g_{kj}\frac{dx^k}{ds}v^j = 0, \tag{1.2}$$

where g_{kj} is the Riemannian metric and s is the arc-length of the curve, maps any geodesic into a geodesic, the equation of Jacobi:

$$\frac{\delta^2 v^h}{ds^2} + R_{kji}{}^h v^k \frac{dx^j}{ds}\frac{dx^i}{ds} = 0 \tag{1.3}$$

is satisfied, where $\frac{\delta}{ds}$ denotes covariant differentiation along the curve, $R_{kji}{}^h$ is the curvature tensor of M and the terms of order higher than one with respect to ε are neglected. If the solution of the equation (1.3) vanishes at a point p_0 and at another point p_1 and if it does not vanish between p_0 and p_1 then the points p_0 and p_1 are said to be conjugate on this geodesic.

Recently K. Yano and I. Mogi studied the distance between consecutive conjugate points on a geodesic in a Kaehlerian manifold and proved the following [2]

THEOREM A. *In a Kaehlerian manifold of positive constant holomorphic curvature k (>0), the distance between two consecutive conjugate points is constant and is given by $2\pi/\sqrt{k}$.*

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On the other hand, a curve $x^i(t)$ in a Kaehlerian manifold defined by

$$\frac{\delta^2 x^h}{dt^2}=\alpha\frac{dx^h}{dt}+\beta\varphi_j{}^h\frac{dx^j}{dt} \tag{1.4}$$

is, by definition [1], a holomorphically planar curve or an h-plane curve, where $\varphi_j{}^h$ is the Kaehlerian structure and α, β are certain functions of t.

The purpose of the present paper is to study an infinitesimal deformation carrying an h-plane curve into a curve of the same kind in a Kaehlerian manifold and to obtain a result analogous to the theorem A on an h-plane curve in a Kaehlerian manifold.

§2. An infinitesimal deformation carrying an h-plane curve into a curve of the same kind in a Kaehlerian manifold.

Let us consider a $2n$-dimensional Kaehlerian manifold with local coordinates $\{x^i\}$. Then the Riemannian metric g_{ji} and the Kaehlerian structure $\varphi_j{}^i$ satisfy the following equations

$$\varphi_i{}^k\varphi_j{}^i=-\delta_j^k,\quad g_{hk}\varphi_j{}^h\varphi_i{}^k=g_{ji},\quad \nabla_k\varphi_j{}^i=0.$$

In a Kaehlerian manifold, we consider a curve $L: x^h=x^h(s)$ parameterized with its arc-length s and satisfies the differential equation

$$\frac{\delta^2 x^h}{ds^2}=a\varphi_j{}^h\frac{dx^j}{ds},\quad (a>0) \tag{2.1}$$

where $\frac{\delta}{ds}$ indicates covariant differentiation along L and a is a constant.

If we use an arbitrary parameter t of L, then the equation (2.1) turns into

$$\frac{\delta^2 x^h}{dt^2}=\alpha\frac{dx^h}{dt}+\beta\varphi_j{}^h\frac{dx^j}{dt}, \tag{2.2}$$

where $\alpha=-\frac{d^2t}{ds^2},\ \beta=a\frac{dt}{ds}$.

Since the integral curve of (2.2) is called a holomorphically planar curve [1], we shall call the integral curve of (2.1) also a holomorphically planar curve or an h-plane curve in a Kaehlerian manifold.

Let v^i be a vector field defined along h-plane curves and assume that for any infinitesimal constant ε, the point transformation:

$$\bar{x}^i=x^i+\varepsilon v^i,\quad g_{kj}v^k\frac{dx^j}{ds}=0 \tag{2.3}$$

maps any h-plane curve into an h-plane curve. Then we say that v^i preserves the h-plane curve.

Now we ask for the condition that v^i preserve the h-plane curve.

By straightforward computations, we have

$$\frac{d^2\bar{x}^h}{ds^2}+\{{}_k{}^h{}_j\}(\bar{x})\frac{d\bar{x}^k}{ds}\frac{d\bar{x}^j}{ds}-a\varphi_j{}^h(\bar{x})\frac{d\bar{x}^j}{ds}$$

$$(2.4)\qquad =(\delta_i^h-\varepsilon v^l\{{}_l{}^h{}_i\})\left(\frac{\delta^2 x^i}{ds^2}-a\varphi_j{}^i\frac{dx^j}{ds}\right)$$

$$+\varepsilon\left[-a\varphi_k{}^h\frac{\delta v^k}{ds}+\frac{\delta^2 v^h}{ds^2}+K_{lkj}{}^h\frac{dx^k}{ds}\frac{dx^j}{ds}v^l\right],$$

where $\{{}_k{}^h{}_j\}$ is the Christoffel symbol, $K_{lkj}{}^h$ is the curvature tensor of the Kaehlerian manifold and terms of order higher than one with respect to ε are neglected.

In the sequel, we always neglect terms of order higher than one with respect to ε.

On the other hand, we get

$$(2.5)\qquad \left(\frac{d\bar{s}}{ds}\right)^2=g_{kj}(\bar{x})\frac{d\bar{x}^k}{ds}\frac{d\bar{x}^j}{ds}=1+2\varepsilon\rho,$$

where we have put

$$(2.6)\qquad \rho=g_{kj}\frac{dx^k}{ds}\frac{\delta v^j}{ds}.$$

Using the relation (2.5), the left member of (2.4) turns into

$$\left(\frac{\delta^2\bar{x}^h}{d\bar{s}^2}-a\varphi_j{}^h\frac{d\bar{x}^j}{d\bar{s}}\right)\left(\frac{d\bar{s}}{ds}\right)^2+\varepsilon\left(\frac{d\rho}{ds}\frac{dx^h}{ds}+a\rho\varphi_j{}^h\frac{dx^j}{ds}\right).$$

Therefore if v^i preserves the h-plane curve then we have

$$(2.7)\qquad \frac{d\rho}{ds}\frac{dx^h}{ds}+a\rho\varphi_j{}^h\frac{dx^j}{ds}=\frac{\delta^2 v^h}{ds^2}+K_{lkj}{}^h v^l\frac{dx^k}{ds}\frac{dx^j}{ds}-a\varphi_j{}^h\frac{\delta v^j}{ds}.$$

From the relation (2.6), we have a system of differential equations along an h-plane curve

$$\rho=a\varphi_{kj}v^k\frac{dx^j}{ds},$$

$$(2.8)\qquad \frac{d\rho}{ds}=a\varphi_{kj}\frac{\delta v^k}{ds}\frac{dx^j}{ds},$$

$$\frac{d^2\rho}{ds^2}=a\left(\varphi_{kj}\frac{\delta^2 v^k}{ds^2}\frac{dx^j}{ds}+a\rho\right).$$

§3. Distance between consecutive conjugate points on an h-plane curve in a Kaehlerian manifold of constant holomorphic curvature.

In this section, we are going to consider an infinitesimal deformation carrying an h-plane curve into a curve of the same kind in a Kaehlerian manifold of positive constant holomorphic curvature k.

In this case, the curvature tensor $K_{lkj}{}^h$ is of the form:

$$K_{lkj}{}^h=\frac{k}{4}(g_{kj}\delta_l^h-g_{lj}\delta_k^h+\varphi_{kj}\varphi_l{}^h-\varphi_{lj}\varphi_k{}^h-2\varphi_{lk}\varphi_j{}^h). \tag{3.1}$$

Substituting (3.1) into (2.7), we obtain

$$\frac{\delta^2 v^h}{ds^2}-a\varphi_j{}^h\frac{\delta v^j}{ds}+\frac{k}{4}v^h=\left[\frac{d\rho}{ds}\delta_j^h+(a+\frac{3}{4}k)\rho\varphi_j{}^h\right]\xi^j, \tag{3.2}$$

where we have put $\frac{dx^j}{ds}=\xi^j$.

If the solution v^h of the equation (3.2) vanishes at a point p_0 and at an another point p_1 and if it does not vanish between p_0 and p_1 then the points p_0 and p_1 are said to be *conjugate* on this h-plane curve.

Taking account of (3.2) and the third relation of (2.8), we get

$$\frac{d^2\rho}{ds^2}=-c\rho, \qquad c=a(a+\frac{3}{4}k)+\frac{k}{4}. \tag{3.3}$$

Consequently above equaton gives

$$\rho=A\sin\sqrt{c}\ s+B\cos\sqrt{c}\ s, \tag{3.4}$$

where A and B are constants.

Now we assume that $v^i=0$ and consequently $\rho=0$ when $s=0$. Then we have

$$\rho=A\sin\sqrt{c}\ s \tag{3.5}$$

from (3.4).

Substituting (3.5) into (3.2), we have

$$\frac{\delta^2 v^h}{ds^2}-a\varphi_j{}^h\frac{\delta v^j}{ds}+\frac{k}{4}v^h=A[\sqrt{c}(\cos\sqrt{c}\ s)\delta_j^h + (a+\frac{3}{4}k)(\sin\sqrt{c}\ s)\varphi_j{}^h]\xi^j, \tag{3.6}$$

A being a constant.

In this place, if we put

$$\frac{\delta v^h}{ds} = q^h, \quad p^h = q^h - b\varphi_j{}^h v^j, \tag{3.7}$$

where b is a non-zero constant given by the relation

$$a - b + \frac{k}{4b} = 0, \tag{3.8}$$

then we easily see that

$$bv^h = \varphi_j{}^h (p^j - q^j). \tag{3.9}$$

Differentiating the second relation of (3.7) and substituting it into (3.6), we obtain

$$\frac{\delta p^h}{ds} + \frac{k}{4b}\varphi_j{}^h p^j = A\left[\sqrt{c}\,(\cos\sqrt{c}\,s)\delta_j^h + (a + \frac{3}{4}k)(\sin\sqrt{c}\,s)\varphi_j{}^h\right]\xi^j, \tag{3.10}$$

by virtue of (3.8).

Regarding (3.10) as a system of simultaneous ordinary differential equations with respect to p^h, there exists a system of solutions $p^h(x(s))$ along an h-plane curve, and moreover this system of solutions is determined uniquely by the system of initial values $p^h(x(0))$ at the point $s=0$ on an h-plane curve.

On the other hand, we can see that

$$p^h = -\frac{A}{a}\left[\frac{k}{4b}(\sin\sqrt{c}\,s)\xi^h + \sqrt{c}\,(\cos\sqrt{c}\,s)\varphi_j{}^h\xi^j\right] \tag{3.11}$$

satisfies the differential equation (3.10) along an h-plane curve.

Therefore under the system of initial conditions

$$p^h(x(0)) = -\frac{A}{a}\sqrt{c}\,(\varphi_j{}^h\xi^j)(x(0)), \tag{3.12}$$

p^h defined by (3.11) is a system of unique solutions of (3.11).

Substituting (3.11) into (3.9) and integrating, we can see that the system of solutions v^h of the system of differential equations (3.6) is determined uniquely by

$$v^h = -\frac{A}{a}(\sin\sqrt{c}\,s)\varphi_j{}^h\xi^j \tag{3.13}$$

under the system of initial conditions

$$v^h(x(0)) = 0, \quad \frac{dv^h}{ds}(x(0)) = -\frac{A}{a}\sqrt{c}\,(\varphi_j{}^h\xi^j)(x(0)). \tag{3.14}$$

From the first equation of (2.8) and (3.5), we can see that, if v^h vanishes at a point, then ρ vanishes at this point and consequently $\sin\sqrt{c}\,s$ vanishes also at this point.

Conversely, from (3.13) we can see that if $\sin\sqrt{c}\,s$ vanishes at a point, then v^h vanishes also this point.

Thus if v^h vanishes at a point $p_0(x^h(0))$, then point at which $\sin\sqrt{c}\,s$ vanishes immediately after $s=0$ is given by $s=\pi/\sqrt{c}$. Thus we have the the following

THEOREM. *In a Kaehlerian manifold of positive constant holomorphic curvature k, the distance between two consecutive conjugate points on an h-plane curve is constant and is given by* $\pi/\sqrt{c}$, *where* $c=a(a+\frac{3}{4}k)+\frac{k}{4}$.

If we consider the case of $a=0$ in (2.1), then the h-plane curve becomes to a geodesic and $\sqrt{c}$ takes the value $\sqrt{k}/2$. Therefore above theorem assures the theorem A stated in §1.

References

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