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On Matroids and Graphs

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1. Introduction

In this paper we show that Eulerian matroids and bipartite matroids are dual concepts. But this does not hold for non-binary matroids.

A *matroid* $M(E, \mathcal{B})$ is a finite set E togather with a non-empty collection of subsets of E, called *bases*, such that

(i) all bases have the same cardinality,

(ii) if B_1 and B_2 are bases, and $x \in B_1$, there exists $y \in B_2$ such that $(B_1 - \{x\}) \cup \{y\}$ is also a base.

A subset of E is an *independent set* if it is contained in some base. Any subset which is not independent is *dependent*. A singleton dependent set is a *loop* and a *circuit* is a minimal dependent set. The collection of all circuits of M is denoted by $\mathcal{C}(M)$. The *dual* M^* of M is the matroid on E consisting of bases of the form E-B where B is a base of M.

A *cocircuit* of $M(E, \mathcal{B})$ is circuit of the dual matroid M^* $(E, \mathcal{B}^*)$ where $\mathcal{B}^* = \{E\text{-}B \mid B \in \mathcal{B}\}$

Let M be a matroid on E and S a subset of E.

We define $M|S$, the *reduction* of M to S, to be the matroid on S whose circuits are those circuits of M contained in S. We define $M \cdot S$, the *contraction* of M to S, to the matroid on S whose circuits are the minimal non-empty members of the collection $\{C \cap S \mid C \in \mathcal{C}(M)\}$.

Generalizing graph-theoretic concepts we define $M(E, \mathcal{B})$ to be an *Euler matroid* if there exist disjoint circuits $C_1, C_2, \ldots\ldots, C_n$ such that

$$E = C_1 \cup C_2 \cup \ldots\ldots \cup C_n.$$

We define $M(E, \mathcal{B})$ to be a *bipartite matroid* if every circuit has even cardinality.

We define M to be a binary matroid if M is representable over the field of integers modulo two. ([1], [2])

2. Lemmas

We shall need the following lemmas.

Lemma 1. *For a matroid M following properties are equivalent:*

ⓐ *M is binary.*

ⓑ *M^* is binary, where M^* is dual matroid of M.*

ⓒ $^{\forall}C \in \mathcal{C}(M)$, $^{\forall}C^* \in \mathcal{C}(M^*)$, $|C \cap C^*| \equiv 0 \ (mod\ 2)$ ([2])

Lemma 2. *Let $M = M(E, \mathscr{B})$ be a matroid. A circuit C^* of M is a minimal non-empty subset of E having a non-empty intersection with every base of M.*

Lemma 3. *If $M(E, \mathscr{B})$ is a binary matroid and E is a subset oj E, then the contraction matroid $M(E', M \cdot E')$ is binary.* ([3])

We use above lemmas to prove the following theorem.

Theorem. *A binary matroid is Euler iff its dual matroid is bipartite.*

3. Proof of theorem

Let $M(E, \mathscr{B})$ be binary and Euler. From the definitions of an Euler matroid, there exist a collection $\{C_1, \ldots\ldots, C_n\}$ of disjoint circuits of $M(E, \mathscr{B})$ such that $E = C_1 \cup C_2 \cup \ldots \cup C_n$. Let C^* be any cocircuit. Then since $M(E, \mathscr{B})$ is binary, by lemma 1 $|C_i \cap C^*| = 2k_i$, $1 \leqq i \leqq n$, where k_i is integer.

Hence $|C^*| = \sum_{i=1}^{n} |C_i \cap C^*| = 2\sum_{i=1}^{n} k_i$ and $M(E, \mathscr{B}^*) = M^*$ is bipartite.

Suppose now $M(E, \mathscr{B})$ is binary and bipartite. We use induction on $|E|$ to show that $M(E, \mathscr{B}^*)$ is Euler. Assume that this holds for matroids having fewer $|E|$ elements.

Let C^* be any cocircuit of $M(E, \mathscr{B})$. We show that there exists such a cocircuit C^*. Let $x \in E$ and suppose x belongs to no cocircuit, then x belongs to every base of $M(E, \mathscr{B}^*)$. (Lemma 2) Hence

$$x \in \bigcap_{B^* \in \mathscr{B}^*} B^* \qquad \therefore \ x \notin \bigcap_{B^* \in \mathscr{B}^*} B$$

Thus $\{x\}$ is a dependent set, so that $\{x\}$ is a loop.

Since $M(E, \mathscr{B})$ is bipartite, a circuit $\{x\}$ has even cardinality and we have arrived at a contradiction.

Next we shall show that S can be expressed is the union of a mutually disjoint set of cocircuits. If C^* is a cocircuit and $E = C^*$, it is trivial. When C^* is proper subset of E, let $E' = E\text{-}C^*$ and consider the contraction matroid $M(E', M \cdot E')$ of $M(E, \mathscr{B})$ to E'. $M(E', M \cdot E')$ has as circuits the minimal non-empty members of the family $\{C \cap E' \mid C \in \mathcal{C}(M)\}$ where $\mathcal{C}(M)$ is the collection of all circuits of $M(E, \mathscr{B})$.

Since $M(E, \mathscr{B})$ is bipartite, $|C|$ is even for all $C \in \mathcal{C}(M)$ and since $M(E, \mathscr{B})$ is binary and C^* is a cocircuit of M, $|C^* \cap C|$ is even for all $C \in \mathcal{C}(M)$. Hence

$$|C \cap E'| = |C \cap (E\text{-}C^*)| = |C| - |C \cap C^*|$$

is even for all $C \in \mathcal{C}(M)$ and therefore the contraction matroid $M(E', M \cdot E')$ is bipartite. By lemma 3, $M(E', M \cdot E')$ is binary. By the induction hypothesis,

$$E' = C_1^* \cup C_2^* \ldots\ldots \cup \cup C_n^*$$

where C_i^* are mutually disjoint cocircuits of $M(E', M\cdot E')$. But any cocircuit of $M(E', M\cdot E')$ isa cocircuit of $M(E, \mathscr{B})$. Hence $E = E' \cup C^* = C_1^* \cup \cdots\cdots \cup C_n^* \cup C^*$ is a partition of E into mutually disjoint cocircuits. Thus $M(E, \mathscr{B}^*)$ is an Euler matroid.

References

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[2] Rabe von Randow(1975) *Introduction to the theory of matroids* Springer-Verlag,

[3] G. J. Minty(1966), On the Axiomatic Foundations of the Theories of Directed Linear Graph, Electrical Networks and Networkprogramming *J. Math. Mech.* 15 485-520

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bipartite graph와 Euler graph의 정의를 사용하는 대신 이들 graph가 나타내는 특성을 사용하여 bipartite matroid와 Euler matroid를 정의하고 이들 matroid가 binary일 때 서로 dual의 관계가 있음을 증명한다. 이 관계를 이용하여 bipartite graph와 Euler graph의 성질을 밝힐 수 있다.