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Prime Z-Filters And Primary Ideals In C(X)

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In this paper, we denote C(X) as the ring of continous real valued functions on a topological space X. If I is an ideal in C(X), the family $Z[I]=\{Z(f); f\in I\}$ is a z-filter on X. If $\mathcal{F}$ is a z-filter on X, then the family $\overleftarrow{Z}[\mathcal{F}]$ is an ideal in C(X). An ideal I is a z-ideal if and only if $Z(f)\in Z[I]$ implies $f\in I$, i.e. $I=\overleftarrow{Z}Z[I]$.

If $\mathcal{F}$ is a z-filter on X, then $\overleftarrow{Z}[\mathcal{F}]$ is a z-ideal. Every z-ideal in C(X) is an intersection of prime ideals. If P is a prime ideal in C(X) then Z[P] is a prime z-filter. A z-filter $\mathcal{F}$ is prime if $Z_1\cup Z_2\in\mathcal{F}$ implies Z_1 or Z_2 is contained in $\mathcal{F}$. An ideal P is primary if $fg\in I$ implies either $f\in I$ or $g^n\in I$ for some $n>0$ and $P\neq C(X)$. If I is primary, $r(I)=\{f\in C(X); f^n\in I$ for some $n>0\}$ is prime. If $\mathcal{F}$ is a prime z-filter, then $\overleftarrow{Z}[\mathcal{F}]$ is prime. [1] [2] [3] [4] [6]

Lemma 1 *For any z-ideal I in C(X), the followings are equivalent:*

(1) *I is primary*

(2) *I contains a primary ideal*

(3) *For all f, $g\in C(X)$, if $fg=0$, then either $f\in I$ or $g^n\in I$ for some $n>0$*

(4) *For every $f\in C(X)$, there is a zero-set in Z[I] on which f does not change sign.* [5]

Theorem 2 *If $\mathcal{J}$ is a primary ideal in C(X), then $Z[\mathcal{J}]$ is a prime z-filter on X.*

Proof. Let $\mathcal{J}'=Z^{\leftarrow}[Z[\mathcal{J}]]$, then $Z[\mathcal{J}']=Z[\mathcal{J}]$ and $\mathcal{J}'$ is a z-ideal which contains the primary ideal $\mathcal{J}$. By Lemma 1 $\mathcal{J}'$ is primary. Now, suppose that $Z(f)\cup Z(g)=Z(fg)\in Z[\mathcal{J}']$, then $fg\in\mathcal{J}'$ since $\mathcal{J}'$ is a primary z-ideal. Hence either $f\in\mathcal{J}'$ or $g^n\in\mathcal{J}'$ for some $n>0$. Thus either $Z(f)\in Z[\mathcal{J}']=Z[\mathcal{J}]$ or $Z(g^n)=Z(g)\in Z[\mathcal{J}']=Z[\mathcal{J}]$. Therefore $Z(f)\in Z[\mathcal{J}]$ or $Z(g)\in Z[\mathcal{J}]$.

Henceforth, there is an one-to-one correspondence between primary ideals in C(X) and prime z-filters on X.

Theorem 3 *If I is a z-ideal, then we have $r(I)=I$, in other words, I is radical.*

Proof. Since every z-ideal I is an intersection of all the prime ideal containing it, thus $r(I)=I$.

Corollary 4 *Every primary z-ideal is prime.*

Theorem 5 *For all ideal I, we have* $Z[I]=Z[R(I)]$

Proof. It is clear that $Z[I]\subset Z[r(I)]$. Conversely, let Z be any zero-set in $Z[r(I)]$. Then there exists $f\in r(I)$ such that $f^n\in I$ for some $n>0$. Hence $Z=Z(f^n)=Z(f)\in Z[I]$, therefore $Z[I]=Z[r(I)]$.

Using Theorem 5 to prove Theorem 2, we can do in another method:

If $\mathcal{J}$ is primary, then $r(\mathcal{J})$ is prime. Hence $Z[\mathcal{J}]=Z[r(\mathcal{J})]$ is a prime z-filter.

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