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On Metrizability of Topological Spaces

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In [1], it was shown that a regular space is metrizable if and only if it is cs-semistratifiable and wΔ-space. Also, in [2], H.R. Bennett and H.W. Martin have shown that a regular space X is a Moore space if and only if X is a c-semistratifiable space and a Moore (mod K) space. And by S.Y. Choi and Y.S. Kim [3], a developable (mod K) T_1-space is wΔ-space.

Here, a cs-stratifiable space will be characterized and will be shown that a regular and compact cs-semistratifiable space is metrizable. In this paper all spaces are assumed to be a T_2-space and all undefined terms and notions may be found in [8].

Definition Let $(X, \mathfrak{T})$ be a topological space and let g be a function from $N \times X$ to $\mathfrak{T}$. Then g is called a *COC-function* for X if it satisfies the following two conditions.:

(i) $x \in \bigcap_{n=1}^{\infty} g(n,x)$ for all $x \in X$,

(ii) $g(n+1,x) \subset g(n,x)$ for all $n \in N$ and $x \in X$.

We adopt the convention that if $\{x_n\}$ is a sequence, $\langle x_n \rangle$ denotes the range of the sequence $\{x_n\}$ and $\langle x; x_n \rangle$ denotes $\{x\} \cup \langle x_n \rangle$.

Definition A *cs-semistratification* for a topological space X is a mapping g from $N \times X$ to the topology of X which satisfies the following conditions:

(i) $x \in g(n,x)$,

(ii) $g(n+1,x) \subset g(n,x)$,

(iii) if a sequence $\{x_n\}$ converges to a unique point x, then

$$\bigcap_{i=1}^{\infty} g(i, <x; x_n>) = <x; x_n>,$$

Here, we use the notation that

$$g(n, s) = \cup \{g(n, s) : s \in S\}$$

for every subset S of X.

A space is said to be *cs-semistratifiable* if X has a cs-semistratification. A cs-semistratification g is a semistratification [6] if g satisfies the following condition:

(*) $F = \cap \{g(n, F) : n = 1, 2, \cdots\}$ for every closed subset F of X.

Theorem 1. *If $g: N \times X \to \mathfrak{T}$ is a COC-function, then the followings are equivalent:*

(1) $\cap g(n, g(n,x)) = \{x\}$.

(2) *If A is a compact subset of X, then* $\cap g(n, A) = A$

(3) *If* $\{x_n\}\to x_0$ *and* $A=\{x_0\}\cup\{x_n : n\in N\}$, *then* $\cap g(n, A)=A$

Proof (1)⇒(2)

Let A be a compact subset of X and $x\in X-A$. Since $\cap g(n,g(n,p))=\{p\}$ for each $p\in X$, there is $m\in N$ such that $g(m,g(m,p))\in X-\{x\}$ for each $p\in A$. Since $\{g(n,p)) : p\in A\}$ is an open covering of A, there are $p_1, p_2, \cdots, p_k$ such that $A\subset g(n_1,p_1)\cup g(n_2,p_2)\cup\cdots\cup g(n_k,p_k)$. Let $n=\max\{n_i : 1\le i\le k\}$. Suppose that there is $p\in A$ such that $x\in g(n,p)$, there is $i(1\le i\le k)$ such that $p\in g(n_i,p_i)$. It follows that $x\in g(n_i,g(n_i,p_i))$. Hence this is contradict to $g(n_i,g(n_i,p_i))=\{p_i\}$. Therefore $x\notin g(n,p)$ and $g(n,A)=\cup\{g(n,a) : a\in A\}=A$.

(2)⇒(3)

Since $A=\{x_0\}\cup\{x_n : n\in N\}$ is a compact, the statement is clear.

(3)⇒(1)

Let $\{g(n,x) : n\in N\}$ be a nested local base at each $x\in X$ and if $\{x_n\}\to x_0$ and $A=\{x_0\}\cup\{x_i : i\in N\}$, then $\cap\{g(n,A) : n\in N\}=A$. Suppose that there are $x, y(\neq)\in X$ such that $y\in\cap\{g(n,g(n,x)) : n\in N\}$, then there exists a sequence $\{x_n\}$ such that $y\in g(n,x_n)$ and $x_n\in g(n,x)$ for each $n\in N$. Then $\{x_n\}\to x$ and there exists a subsequence $\{x_{i_n}\}$ converging to x and having no term equal to y. Set $A=\{x\}\cup\{x_{i_n} : n\in N\}$. Then there exists $n\in N$ such that $y\notin g(n,A)$. In particular, since $g(i_n,x_{i_n})\subset g(i_n,A)\subset g(n,A)$, $y\in(i_n,x_{i_n})$. Therefore $y=x$ and hence $\cap\{g(n,g(n,x)) : n\in N\}=\{x\}$.

Definition A topological space X is *developable(mod K)* if $\mathcal{G}=\{\mathcal{G}_i : i\in N\}$ where $\mathcal{G}_i$ is an open covering of X for each natural number i and for each $x\in X$, if $x\in K\in\mathcal{H}$ (where $\mathcal{H}$ is a compact covering) and K is contained in an open set V, then there is a natural number $n(x)$ such that $st(x, \mathcal{G}_{n(x)})\subset V$. A regular developable (mod K) space is called a *Moore (mod K)* space and $\mathcal{G}$ is called a *development (mod K)* for X.

Definition A topological space X is a $w\Delta$-space if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \cdots$ of open coverings of X such that, for each x in X, if $x_n\in st(x, \mathcal{G}_n)$ for $n=1,2,\cdots$, then the sequence $\{x_n\}$ has a cluster point.

Theorem 2. *Let X be a regular space. If X is a developable(mod K) space, then X is a $w\Delta$-space.*

Using the above theorem, the following theorem can be derived.

Theorem 3. *In a regular space X, the followings are equivalent:*

(1) *X is a developable space*

(2) *X is a developable(mod K) and cs-semistratifiable space.*

Proof (1)⇒(2) clear.

(2)⇒(1) Since a regular developable(mod K) space is a wΔ-space, a regular cs-semistratifiable wΔ-space is a developable space from [1].

Corollary 4. *A regular compact cs-semistratifiable space is metrizable.*

Proof Since a compact space is a developable(mod K) space and a cs-semistratifiable and developable(mod K) space is a developable space. From [7], it is clear that X is metrizable.

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