韓國數學敎育學會誌〈數學敎育〉
1978. 12. Vol. XVII No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1978. Vol. XVII No. 1

A Characterization of Valuation Domain

By J. W. Nam, C. K. Bae, B. H. Park, K. J. Min

Gyeongsang National University, Jinju, Korea

1. Introduction

In this paper, we have a characterization of valuation domain with the pseudo valuation domain closely related to valuation ring and this is motivated in [4]. There are many characterization of valuation domain in [5] and there, the end of the book, is an example (almost Dedekind domain) of Prufer ring that are not Dedekind domain.

2. Definitions and Properties

Definition 2.1 Let R be a domain with quotient field K. A prime ideal P of R is called *strongly prime* if $x, y \in K$ and $xy \in P$ imply that $x \in P$ or $y \in P$.

Definition 2.2 A domain R is called a *pseudo valuation domain* if every prime ideal of R is strongly prime.

Definition 2.3 An integral domain R is a *GCD domain* if any two elements in R have a greatest common divisor.

Proposition 2.4 *Every valuation domain is a pseudo valuation.*

Proof. Let V be a valuation domain, and let P be a prime ideal in V. Suppose $xy \in P$ where $x, y \in K$, the quotient field of V. If both x and y are in V, we are done. Suppose that $x \notin V$. Since V is a valuation domain, we have $x^{-1} \in V$. Hence $y = xy \cdot x^{-1} \in P$, as desired.

Proposition 2.5 *In a pseudo valuation domain R, the prime ideals are linearly ordered. In particular R is quasi local.*

Proof. Let P and Q be prime ideals, and suppose $a \in P - Q$. Then for each $b \in Q$ we have $a/b \in R$. Hence $(b/a)P \subset P$ by the proposition. Thus $b = (b/a)a \in P$ and we have $Q \subset P$.

Proposition 2.6 *A GCD domain is integrally closed.*

Proof. [1. Thm 50]

Corollary 2.7 *Every valuation domain is GCD domain.*

Proof. Trivial.

3. Main Theorem

Theorem 3.1 *The following statements are equivalent.*

(1) *R is a pseudo valuation domain and GCD domain.*

(2) *R is an integrally closed quasi local domain whose primes are linearly ordered by inclusion and the intersection of any two principal ideals is finitely generated.*

(3) *R is valuation domain.*

Proof. (1)⇨(2) Trivial by Proposition 2.5 and 2.6.

(2)⇨(3) [2. Thm 1]

(3)⇨(1) By Prop. 2.4 and Cor. 2.7.

Even though R is a valuation domain but has not a nonzero principal prime ideal always in pseudo valuation domain. Since discrete rank 1 valuation ring has no proper prime ideal but the converse (A pseudo valuation domain R has a nonzero principal prime ideal, then R is valuation domain) is true.

Thus this condition is stronger than the condition of valuation domain.

Corollary 3.2 *If R is a pseudo valuation domain and GCD domain, then R is Prufer domain.*

Proof. By hypothesis R is valuation domain thus Prufer domain.

Example 3.3 Let m be a square free positive integer $m \equiv 5 \pmod 8$. Let Z denote the ring of integers and set $D=Z[\sqrt{m}]$. Since $m \equiv 1 \pmod 4$, D does not contain the algebraic integers of the form $(a+b[\sqrt{m})/2$, where a and b are odd integer, thus D is not integrally closed. It is routine to check that $(2, 1+\sqrt{m})=N$ is a maximal ideal of D. The desired example is $R=D_N$, which has $K=Q[\sqrt{m}]$ as its quotient field. R is not a valuation ring since neither $(1+\sqrt{m})/2$ nor its inverse lies in R.

In above example R is pseudo valuation domain.

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