韓國數學敎育學會誌〈數學敎育〉
1978. 12 Vol. XVII No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1978. Vol. XVII No. 1

On the Relationship between Two Connections in n-dimensional *g-UFT

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I. Introduction

A. n-dimensional $*g^{\lambda\nu}$-unified field theory

In the usual Einstein's unified field theory the generalized n-dimensional Riemannian space X_n is endowed with a real non-symmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and its skew-symmetric part $k_{\lambda\mu}$:

(1.1) $$g_{\lambda\mu}=h_{\lambda\mu}+k_{\lambda\mu},$$

where

$$\mathrm{Det}(g_{\lambda\mu})\neq 0,\quad \mathrm{Det}(h_{\lambda\mu})\neq 0.$$

On the other hand, *Einstein's $*g^{\lambda\nu}$-unified field theory ($*g$-UFT) in the same space X_n is defined to be based upon the basic real tensor $*g^{\lambda\nu}$ defined by*

(1.2) $$g_{\lambda\mu}{*g^{\lambda\nu}}=\delta_\mu^\nu.$$

It may also be decomposed into its symmetric part $*h^{\lambda\nu}$ and its skew-symmetric part $*k^{\lambda\nu}$:

(1.3) $$*g^{\lambda\nu}=*h^{\lambda\nu}+*k^{\lambda\nu}.$$

Since Det $(*h^{\lambda\nu})\neq 0$, we may define an unique tensor $*h_{\lambda\mu}$ by

(1.4) $$*h_{\lambda\mu}{*h^{\lambda\nu}}=\delta_\mu^\nu.$$

In our $*g$-UFT we use both $*h_{\lambda\mu}$ and $*h^{\lambda\nu}$, instead of $h_{\lambda\mu}$ and $h^{\lambda\nu}$, as the tensors for raising and/or lowering indices of all tensors defined in X_n in the usual manner, with the exception of the tensors $g_{\lambda\mu}$, $h_{\lambda\mu}$, and $k_{\lambda\mu}$ in order to avoid the notational confusion. We then have, for example,

(1.5) a $$*k_{\lambda\mu}=*k^{\rho\sigma}{*h_{\lambda\rho}}{*h_{\mu\sigma}},\quad *g_{\lambda\mu}=*g^{\rho\sigma}{*h_{\lambda\rho}}{*h_{\mu\sigma}},$$

so that

(1.5) b $$*g_{\lambda\mu}=*h_{\lambda\mu}+*k_{\lambda\mu}.$$

The basic connection $\Gamma^\nu_{\lambda\mu}$ in our $*g$-UFT is given by the following Einstein's equation expressed in terms of $*g^{\lambda\nu}$:

(1.6) a $$\partial_\omega{*g^{\lambda\mu}}+\Gamma^\lambda_{\alpha\omega}{*g^{\alpha\mu}}+\Gamma^\mu_{\omega\alpha}{*g^{\lambda\alpha}}=0,$$

or equivalently

(1.6) b $$D_\omega{*g^{\lambda\mu}}=-2S_{\omega\alpha}{}^{\mu}{*g^{\lambda\alpha}},$$

where D_ω is the symbolic vector of the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$ and

(1.7) $$S_{\lambda\mu}{}^{\nu}(*g^{\lambda\nu})\stackrel{\mathrm{df}}{=}\Gamma^\nu_{[\lambda\mu]}.$$

For the convenience of writing we shall make the following agreement in our further considerations:

Agreement (1.1). *We omit the notation "*" from all starred tensors and operators in the remaining of the present paper, and understand that they are starred.*

B. The problem to be solved

As givenin (1.3) and (1.6), $*g$-UFT is based on a real tensor

$$g^{\lambda\nu}=h^{\lambda\nu}+k^{\lambda\nu} \tag{1.3}$$

and a real connection $\Gamma^{\nu}_{\lambda\mu}$ which satisfies the field equation

$$\partial_{\omega}g^{\lambda\mu}+\Gamma^{\lambda}_{\alpha\omega}g^{\alpha\mu}+\Gamma^{\mu}_{\omega\alpha}g^{\lambda\alpha}=0. \tag{1.6}$$

Consider another unified field theory based on a real tensor

$$\bar{g}^{\lambda\nu}\overset{\mathrm{df}}{=}\bar{h}^{\lambda\nu}+\bar{k}^{\lambda\nu}\overset{\mathrm{df}}{=}h^{\lambda\nu}-k^{\lambda\nu};\quad \bar{h}^{\lambda\nu}=\bar{h}^{(\lambda\nu)},\quad \bar{k}^{\lambda\nu}=\bar{k}^{[\lambda\nu]}, \tag{1.8}$$

and a real connection $\bar{\Gamma}^{\nu}_{\lambda\mu}$ which satisfies

$$\partial_{\omega}\bar{g}^{\lambda\mu}+\bar{\Gamma}^{\lambda}_{\alpha\omega}\bar{g}^{\alpha\mu}+\bar{\Gamma}^{\mu}_{\omega\alpha}\bar{g}^{\lambda\alpha}=0. \tag{1.9}$$

The main purpose of the present paper is to find the relationship between $\Gamma^{\nu}_{\lambda\mu}$ given by (1.6) and $\bar{\Gamma}^{\nu}_{\lambda\mu}$ given by (1.9).

II. Preliminary results

In this chapter some results which are essential to the present paper will be briefly stated without proofs. The detailed proofs are given in [1].

The following tensors will be used in our further considerations:

(2.1) a $$^{(0)}k_{\lambda}{}^{\nu}\overset{\mathrm{df}}{=}\delta_{\lambda}^{\nu},\quad ^{(1)}k_{\lambda}{}^{\nu}\overset{\mathrm{df}}{=}k_{\lambda}{}^{\nu},\quad ^{(p)}k_{\lambda}{}^{\nu}\overset{\mathrm{df}}{=}{}^{(p-1)}k_{\lambda}{}^{\alpha}k_{\alpha}{}^{\nu}.$$

(2.1) b $$\overset{pqr}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu}\overset{\mathrm{df}}{=}{}^{(p)}k_{\omega}{}^{\alpha\,(q)}k_{\mu}{}^{\beta\,(r)}k_{\nu}{}^{\gamma},$$

(2.1) c $$2\overset{(pq)r}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu}\overset{\mathrm{df}}{=}\overset{qpr}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu}+\overset{pqr}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu},\quad 2\overset{[pq]r}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu}\overset{\mathrm{df}}{=}\overset{pqr}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu}-\overset{qpr}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu}.$$

Also, denoting the tensor $T_{\omega\mu\nu}$ by T, the following abbreviations will be used:

(2.2) a $$\overset{pqr}{T}\overset{\mathrm{df}}{=}\overset{pqr}{T}_{\omega\mu\nu}\overset{\mathrm{df}}{=}\overset{pqr}{A}{}^{\alpha\beta\gamma}_{\omega\mu\nu}T_{\alpha\beta\gamma},$$

(2.2) b $$T\overset{\mathrm{df}}{=}T_{\omega\mu\nu}\overset{\mathrm{df}}{=}\overset{000}{T}.$$

The connection $\Gamma^{\nu}_{\lambda\mu}$ defined by (1.6) is given by

$$\Gamma^{\nu}_{\lambda\mu}=\left\{{\nu \atop \lambda\mu}\right\}+S_{\lambda\mu}{}^{\nu}+U_{\lambda\mu}{}^{\nu} \tag{2.3}$$

where $\left\{{\nu \atop \lambda\mu}\right\}$ are the Christoffel symbols defined by $h_{\lambda\mu}$ in the usual way and

$$U_{\nu\lambda\mu}\overset{\mathrm{df}}{=}\overset{(10)0}{S}_{(\lambda\mu)\nu}+2\overset{(10)0}{S}_{\nu(\lambda\mu)}. \tag{2.4}$$

The system of equations (1.6) is equaivalent to

$$S_{\alpha\beta\gamma}X^{\alpha\beta\gamma}_{\omega\mu\nu}=B_{\alpha\beta\gamma}, \tag{2.5}$$

where

$$X^{\alpha\beta\gamma}_{\omega\mu\nu}\overset{\mathrm{df}}{=}\overset{000}{A}{}^{(\alpha\beta)\gamma}_{(\omega\mu)\nu}+\overset{110}{A}{}^{(\alpha\beta)\gamma}_{(\omega\mu)\nu}+2\overset{(10)1}{A}{}^{(\alpha\beta)\gamma}_{(\omega\mu)\nu}, \tag{2.6}$$

$$B_{\omega\mu\nu}\overset{\mathrm{df}}{=}\tfrac{1}{2}(K_{\omega\mu\nu}+3K_{\alpha(\mu\beta}k_{\omega)}{}^{\alpha}k_{\nu}{}^{\beta}), \tag{2.7}$$

$$K_{\omega\mu\nu}\overset{\mathrm{df}}{=}\nabla_{\omega}k_{\lambda\mu}+\nabla_{\mu}k_{\omega\lambda}+\nabla_{\lambda}k_{\omega\mu}. \tag{2.8}$$

Here ∇_{ω} is the symbolic vector of the covariant derivative with respect to $\left\{{\nu \atop \lambda\mu}\right\}$.

III. Main theorem

The first consequences of (1.3), (1.8), (2.6), (2.7), and (2.8) are formally stated in the following two lemmas:

Lemma (3.1). *We have*

(3.1) *a* $\bar{h}^{\lambda\nu}=\bar{h}^{\lambda\nu},\ \bar{h}_{\lambda\mu}=h_{\lambda\mu},$

(3.1) *b* $\bar{k}^{\lambda\nu}=-k^{\lambda\nu},\ \bar{k}_{\lambda\mu}=-k_{\lambda\mu},\ \bar{k}_{\lambda}{}^{\nu}=-k_{\lambda}{}^{\nu},$

(3.1) *c* $\bar{g}^{\lambda\nu}=g^{\nu\lambda},\ \bar{g}_{\lambda\mu}=g_{\mu\lambda}.$

Lemma(3.2). *We have*

(3.2) *a* $\bar{K}_{\nu\omega\mu}=-K_{\omega\mu\nu},$

(3.2) *b* $\bar{B}_{\omega\mu\nu}=-B_{\omega\mu\nu},$

(3.2) *c* $\bar{X}^{\alpha\beta\gamma}_{\omega\mu\nu}=X^{\alpha\beta\gamma}_{\omega\mu\nu}.$

Proof. (3.1) are direct consequences of the definition of $\bar{g}^{\lambda\mu}$. (3.2) follow from (2.6), (2.7), and (2.8) using (2.1) and (3.1)b.

Finally, we have the following main theorem which states that the connection $\bar{\Gamma}^{\nu}_{\lambda\mu}$ is pseudo-Hermitian symmetric in the inidces λ, μ.

Theorem (3.3). *If the system* (1.6) *admits aunique solution, then we have the following relation:*

(3.3) $$\bar{\Gamma}^{\nu}_{\lambda\mu}=\Gamma^{\nu}_{\lambda\mu}.$$

Proof. Assume that (1.6) has a unique solution. Then $\Gamma^{\nu}_{\lambda\mu}$ is expressed as a function of $g^{\lambda\nu}$. Observing (1.6) and (1.9), we see that $\bar{\Gamma}^{\nu}_{\lambda\mu}$ is the same function of $\bar{g}^{\lambda\nu}$. Hence, comparing (2.5) with

(3.4) $$\bar{S}_{\alpha\beta\gamma}\bar{X}^{\alpha\beta\gamma}_{\omega\mu\nu}=\bar{B}_{\omega\mu\nu},$$

which is equivalent to (1.9), and using (3.2), we have

(3.5) $$\bar{S}_{\omega\mu\nu}=-S_{\omega\mu\nu}=S_{\omega\mu\nu},$$

so that

(3.6) $$\bar{U}_{\nu\lambda\mu}=U_{\nu\lambda\mu}=U_{\nu\mu\lambda}$$

according to (2.1), (2.2), (2.4), (3.1), and (3.5). On the other hand, we have

(3.7) $$\overline{\left\{{\nu \atop \lambda\mu}\right\}}=\left\{{\nu \atop \lambda\mu}\right\}=\left\{{\nu \atop \mu\lambda}\right\}$$

in virtue of (3.1)a. Hence, according to (2.3),

$$\bar{\Gamma}^{\nu}_{\lambda\mu}=\overline{\left\{{\nu \atop \lambda\mu}\right\}}+\bar{S}_{\lambda\mu}{}^{\nu}+\bar{U}_{\lambda\mu}{}^{\nu}=\left\{{\nu \atop \mu\lambda}\right\}+S_{\mu\lambda}{}^{\nu}+U_{\mu\lambda}{}^{\nu}=\Gamma^{\nu}_{\mu\lambda}.$$

References

[1] K.T. Chung (1963), Einstein's connection in terms of $*g^{\lambda\nu}$, *Il Nuovo Cimento*, (X)27, 1297-1324.

[2] V. Hlavatý (1957), *Geometry of Einstein's unified field theory*, P. Noordhoff Ltd, Groningen.