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On the fundamental equations of surface theory

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1. Introduction

The formalism and derivation of the fundamental equations of the surface theory can be greatly simplified with the use of the tensors and tensor notation. However, this will require some knowledge of tensor analysis.

The purpose of the present paper is to treat and derive the fundamental equations of the surface theory using both vectorial method and a little tensorial notations. No new results are seeked in this paper, but the method and proofs employed and presented here are more refined, simpler, and somewhat different from the earlier works on the basis that one uses a little knowledge of tensorial notations.

2. Induced Metricc of the Surface

A surface S may be regarded as a two-dimensional submanifold of there-dimensional Euclidean space E_3. The C^∞map $\phi : S \longrightarrow E_3$, defined by

(2.1) $$x^\alpha = x^\alpha(u^1, u^2) \quad (\alpha = 1, 2, 3),$$

relates the parameters (u^1, u^2) of a point P of S to the rectangular Cartesian coordinates (x^α) of P referred to a suitable set of axes in E_3. (2.1) is precisely the parametric form of the equations of S.

Agreement. In the present paper, Greek indices are used for the range 1, 2, 3, and Roman indices for the range 1, 2. Both indices are understood to follow the summation convention.

Let

(2.2) $$g_{ij} \stackrel{\text{def}}{=} \mathbf{x}_i \cdot \mathbf{x}_j,$$

where

$$\mathbf{x}_i = \frac{\partial \mathbf{x}}{\partial u^i}, \quad \mathbf{x}_{ij} = \frac{\partial^2 \mathbf{x}}{\partial u^i \partial u^j}.$$

Theorem (2.1). g_{ij} *are the components of the induced metric tensor of* S.

Proof. According to the transformation of parameters

(2.3) $$u^i = u^i(\bar{u}^1, \bar{u}^2) \qquad \left(\left|\frac{\partial u}{\partial \bar{u}}\right| \neq 0\right),$$

we have

$$\bar{g}_{ij} = \mathbf{x}_{\bar{i}} \cdot \mathbf{x}_{\bar{j}} = \left(\mathbf{x}_a \frac{\partial u^a}{\partial \bar{u}^i}\right) \cdot \left(\mathbf{x}_b \frac{\partial u^b}{\partial \bar{u}^j}\right) = g_{ab} \frac{\partial u^a}{\partial \bar{u}^i} \frac{\partial u^b}{\partial \bar{u}^j},$$

which together with the definition of g_{ij} shows that g_{ij} are components of a symmetric tensor. On the other hand, since the displacement vector dx tangential to S is

$$dx = x_1 du^1 + x_2 du^2,$$

the element of length ds is given by

$$ds^2 = dx \cdot dx = g_{ij} du^i du^j. \tag{2.5}$$

The positive definiteness of the form (2.5) implies that

$$g \overset{\text{def}}{=} |g_{ij}| \neq 0. \tag{2.6}$$

Our assertion follows from (2.4), (2.5), and (2.6).

Remark (2.2). Theorem (2.1) is a natural result from the view-point that S is a two-dimensional submanifold $V_2(g_{ij}; u^i)$ immersed in a Riemannian E_3 $(\delta_{\alpha\beta}; x^\alpha)$ and that g_{ij} and $\delta_{\alpha\beta}$ are related by $g_{ij} = \delta_{\alpha\beta} \frac{\partial x^\alpha}{\partial u^i} \frac{\partial x^\beta}{\partial u^j}$, which is another form of (2.2).

It is well known that the quantities

$$g^{ij} \overset{\text{def}}{=} \frac{\text{cofactor of } g_{ij} \text{ in } g}{g} \tag{2.7}$$

are components of a symmetric contravariant tensor and satisfy

$$g_{ij} g^{ik} = \delta_j^k. \tag{2.8}$$

We may use both g_{ij} and g^{ij} as induced metric tensors of S and employ them as tensors raising and/or lowering the indices of quantities defined in S.

The quantities Ω_{ij}, defined by

$$\Omega_{ij} \overset{\text{def}}{=} N \cdot x_{ij} \qquad \text{(N: Surface normal of S)} \tag{2.9}$$

are obviously the coefficients of the second fundamental form of S.

Theorem (2.3). *Ω_{ij} are the components of a symmetric tensor.*

Proof. Since $x_i \cdot N = 0$, we have according to (2.3)

$$\bar{\Omega}_{ij} = x_{\bar{i}\bar{j}} \cdot N = \frac{\partial}{\partial \bar{u}^i}\left(x_k \frac{\partial u^k}{\partial \bar{u}^j}\right) \cdot N = \left(x_{km} \frac{\partial u^k}{\partial \bar{u}^j} \frac{\partial u^m}{\partial \bar{u}^i} + x_k \frac{\partial^2 u^k}{\partial \bar{u}^i \partial \bar{u}^j}\right) \cdot N = \Omega_{km} \frac{\partial u^k}{\partial \bar{u}^j} \frac{\partial u^m}{\partial \bar{u}^i},$$

which proves our assertion.

3. Fundamental Equations of Surface Theory

In the present section, we shall prove the fundamental equations of surface theory; namely Gauss' equations, the equations of Weingarten, the equation of Gauss, and the equation of Codazzi.

Firet of all, several notations are introduced. The quantities

$$\Gamma_{ijk} \overset{\text{def}}{=} x_{ij} \cdot x_k \tag{3.1 a}$$

are called the Christoffel symbols of the first kind, and quantities

$$\Gamma_{ij}^k \overset{\text{def}}{=} \Gamma_{ijm} g^{mk} \tag{3.1 b}$$

the Christoffel symbols of the second kind. Using (2.8) we may easily have

$$\Gamma_{ijk} = \Gamma_{ij}^m g_{mk}. \tag{3.1 c}$$

Theorem (3.1). *We have*

$$\Gamma_{ijk}=\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^i}+\frac{\partial g_{jk}}{\partial u^j}-\frac{\partial g_{ij}}{\partial u^k}\right).$$

Proof. The assertion follows easily from (2.2) and (3.1) a.

Riemann symbols of the second kind and of the first kind are defined, respectively, as

(3.3) a $$R^h{}_{ijk}\overset{\text{def}}{=}\frac{\partial}{\partial u^j}\Gamma^h_{ik}-\frac{\partial}{\partial u^k}\Gamma^h_{ij}+\Gamma^m_{ik}\Gamma^h_{mj}-\Gamma^m_{ij}\Gamma^h_{mk},$$

(3.3) b $$R_{hijk}\overset{\text{def}}{=}g_{mh}R^m{}_{ijk}.$$

We know that they are components of tensors.

Now we are ready to prove our main theorem.

Theorem (3.2). *We have*

(*a*) *Gauss's equations*

(3.4) *a* $$x_{ij}=\Gamma^k_{ij}x_k+\Omega_{ij}N$$

(*b*) *Equations of Weingarten*

(3.4) *b* $$N_i=-\Omega_{ij}g^{jk}x_k$$

(*c*) *Equation of Gauss*

(3.4) *c* $$R_{hijk}=\Omega_{hj}\Omega_{ik}-\Omega_{hk}\Omega_{ij}$$

(*d*) *Equations of Codazzi*

$$\frac{\partial\Omega_{ij}}{\partial u^k}-\frac{\partial\Omega_{ik}}{\partial u^j}=\Gamma^m_{ik}\Omega_{mj}-\Gamma^m_{jk}\Omega_{mk}.$$

Proof of (a)

Since x_1, x_2, and N are basis on S, we may put

$$x_{ij}=A^k_{ij}x_k+B_{ij}N.$$

Multiplying N scalarly, we have

$$B_{ij}=x_{ij}\cdot N=\Omega_{ij}.$$

Multiplying x_m scalarly again, we have

$$x_{ij}\cdot x_m=A^k_{ij}(x_k\cdot x_m)\Longrightarrow\Gamma_{ijm}=A^k_{ij}g_{km}\Longrightarrow A^h_{ij}=\Gamma^h_{ij}.$$

These complete our proof of (3.4) a.

Proof of (b)

Since $N\cdot N=1$, we have

$$N\cdot N_i=0\Longrightarrow N_i=b^k_i x_k\Longrightarrow N_i\cdot x_j=b^k_i(x_k\cdot x_j)\Longrightarrow b^k_i g_{kj}=-\Omega_{ij}.$$

In the last step, we used the fact that $N\cdot x_j=0$ implies $N_i\cdot x_j+N\cdot x_{ij}=0$. Now, multiplying both sides of the last result by g^{jm} and using (2.8), we have

$$b^m_i=-\Omega_{ij}g^{jm},$$

which proves our assertion (3.4) b.

Proof of (3.4) c, d

Differentiating both sides of (3.4) a with respect to u^k and using (3.4)a, b, we have

(*) $$x_{ijk}=\frac{\partial}{\partial u^k}\Gamma^n_{ij}x_n+\Gamma^n_{ij}(\Gamma^m_{nk}x_m+\Omega_{nk}N)+\frac{\partial\Omega_{ij}}{\partial u^k}N-\Omega_{ij}\Omega_{kn}g^{nm}x_m.$$

If we subtract (*) from the similar equation for x_{ikj}, which may be obtained from (*) by interchanging j and k throughout, and eliminate x_{ijk}, the resulting equation is reducible by means of (3.3) a to

$$(**)\qquad \mathbf{x}_m(R^m{}_{ijk}-\Omega_{ik}\Omega_{jn}g^{nm}+\Omega_{ij}\Omega_{kn}g^{nm})-N\left(\frac{\partial\Omega_{ik}}{\partial u^j}-\frac{\partial\Omega_{ij}}{\partial u^k}+\Gamma^m_{ik}\Omega_{mj}-\Gamma^m_{ij}\Omega_{mk}\right)=0.$$

Since $\mathbf{x}_m$ and **N** are linearly independent, the coefficients of (**) are all zero. Hence we have proved (3.4) c, d.

Remark (3.3). According to Gauss's notation

$$g_{11}=E,\ g_{12}=g_{21}=F,\ g_{22}=G;$$

$$(3.5)\qquad g^{11}=\frac{G}{g},\ g^{12}=g^{21}=-\frac{F}{g},\ g^{22}=\frac{E}{g}\quad (g=EG-F^2);$$

$$\Omega_{11}=e,\ \Omega_{12}=\Omega_{21}=f,\ \Omega_{22}=g.$$

Using (3.5), the fundamental equations can be reduced to the familiar form

(a) Gauss's equations

$$(3.6)\ \mathrm{a}\qquad \begin{aligned}\mathbf{x}_{11}&=\Gamma^1_{11}\mathbf{x}_1+\Gamma^2_{11}\mathbf{x}_2+eN\\ \mathbf{x}_{12}&=\Gamma^1_{12}\mathbf{x}_1+\Gamma^2_{12}\mathbf{x}_2+fN\\ \mathbf{x}_{22}&=\Gamma^1_{22}\mathbf{x}_1+\Gamma^2_{22}\mathbf{x}_2+gN\end{aligned}$$

(b) Weingarten's equations

$$(3.6)\ \mathrm{b}\qquad \begin{aligned}N_1&=\frac{fF-eG}{g}\mathbf{x}_1+\frac{eF-fF}{g}\mathbf{x}_2\\ N_2&=\frac{gF-fG}{g}\mathbf{x}_1+\frac{fF-gE}{g}\mathbf{x}_2\end{aligned}$$

(c) Equation of Gauss

$$(3.6)\ \mathrm{c}\qquad R_{1212}=eg-f^2$$

(d) Equations of Codazzi

$$(3.6)\ \mathrm{d}\qquad \begin{aligned}\frac{\partial e}{\partial u^2}-\frac{\partial f}{\partial u^1}&=e\Gamma^1_{12}+f(\Gamma^2_{12}-\Gamma^1_{11})-g\Gamma^2_{11}\\ \frac{\partial f}{\partial u^2}-\frac{\partial g}{\partial u^1}&=e\Gamma^2_{22}+f(\Gamma^2_{22}-\Gamma^1_{12})-g\Gamma^2_{12}.\end{aligned}$$

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