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A Study on Two-norm Spaces

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The theory of two-norm spaces is a relatively new invention of the Polish school. It has been applied to smmability theory successfully. The purpose of this paper is to study some basic properties of two-norm spaces and Saks spaces, and apply the result to the theory of Banach spaces.

We begin with some definitions given in [1], [2], and [3].

A Frechet norm(F-norm) $|\ \ |$ on a linear set X is a real valued nonnegative function with the following properties:

(1) $|x|=0$ if and only if $x=0$,

(2) $|x+y|\leqq|x|+|y|$ for all x, y in X,

(3) if $\{a_n\}$ is a sequence of real numbers converging to a real number a and $\{x_n\}$ is a sequence of points of X with $|x_n-x|\to 0$, then $|a_nx_n-ax|\to 0$.

A B-norm for a linear set X satisfies (1) and (2) of the above definition. But, condition (3) is replaced by

(4) $|ax|=|a||x|$, where a is any real number and x is any element of X.

Let $|\ \ |_1$ and $|\ \ |_2$ be two (B- or F-)norms defined on X. We define

$$|\ \ |_1\geqq|\ \ |_2$$

if $|x_n|_1\to 0$ implies $|x_n|_2\to 0$.

When $|\ \ |_1\geqq|\ \ |_2$ and $|\ \ |_2\geqq|\ \ |_1$, we say that $|\ \ |_1$ is equivalent to $|\ \ |_2$ and write $|\ \ |_1\sim|\ \ |_2$.

A two-norm space is a linear set X with two norms, a B-norm $|\ \ |_1$ and an F-norm $|\ \ |_2$. A sequence $\{x_n\}$ of points in a two-norm space $(X, |\ \ |_1, |\ \ |_2)$ is said to be γ-convergent to x in X, written $x_n\xrightarrow{\gamma}x$, if

$$\lim\sup_n|x_n|_1<\infty \text{ and } \lim_n|x_n-x|_2=0.$$

A sequence $\{x_n\}$ in a two-norm space is said to be γ-Cauchy if

$$(x_{p_n}-x_{q_n})\longrightarrow 0 \text{ as } p_n, q_n\longrightarrow\infty.$$

A two-norm space $X_s=(X, |\ \ |_1, |\ \ |_2)$ is called γ-complete if for every γ-Cauchy sequence $\{x_n\}$ in X_s there exists an x in X_s such that $x_n\xrightarrow{\gamma}x$.

A γ-linear functional f on a two-norm space is a real valued function on X_s such that

(1) $f(ax+by)=af(x)+bf(y)$, for all real numbers a, b and any x, y in X_s,

(2) if $x_n \xrightarrow{\gamma} x$, then $f(x_n) \longrightarrow f(x)$.

The set of all γ-linear functionals on X_s will be denoted by X_s^*. It is easy to see that X_s^* is a linear set.

Let X be a linear set and suppose that $|\ |_1$ is a B-norm, and $|\ |^*$ is an F-norm on X. Let $X_s=\{x \in X: |x|_1 \leqq 1\}$ and define

$$d(x,y)=|x-y|^* \text{ for } x, y \text{ in } X_s.$$

Then d is a metric on X_s and the metric space (X_s, d) will be called a Saks set. If (X_s, d) is complete it will be called a Saks space. We shall denote (X_s, d) by $(X, |\ |_1, |\ |^*)$.

A linear functional L_x on $(X_s^*, |\ |_1^*)$ is defined by $L_x(f)=f(x)$ for each f in X_s^*.

In many cases, it makes no difference whether we work in the setting of Saks sets or that of two-norm spaces. An advantage of working in the setting of a Saks set is that we may use category arguments. A disadvantage is that a Saks set is not a linear set while a two-norm space is a linear set.

We now state a theorem which tells us when weak convergence is equivalent to convergence in the strong topology.

Theorem. *Let* $(X, |\ |_X)$ *be a Banach space and let* $(Y, |\ |_Y)=(X, |\ |_X)^*$. *Suppose that there exists a Frechet norm* $|\ |_F$ *such that the Saks set* $Y_s=(Y, |\ |_Y, |\ |_F)$ *is compact and satisfies* (Σ) (see [4]). *If* Y_s^* *contains* $\{L_X : x \in X\}$, *then weak convergence is equivalent to convergence in the strong topology in* $(X, |\ |_X)$.

Proof. Let $\{x_x\}$ be a sequence in X converging to zero. Then for any given positive real number ε there exists an integer $N(\varepsilon)$ such that

$$|L_{x_n}(y)|<\varepsilon \text{ for all } n<N,$$

for each $y \in Y_s=\{y \in Y: |y|_Y \leqq 1\}$.

Since $x_n \longrightarrow 0$ weakly in X, $y(x_n) \longrightarrow 0$ for each $y \in Y$. Thus

$$L_{x_n}(y)=y(x_n) \longrightarrow 0 \text{ for each } y \in Y.$$

Since each $L_{x_n} \in Y_s^*$ and Y_s is complete and satisfies (Σ), $\{L_{x_n}\}$ is equicontinuous at zero in Y_s. Hence given any positive real number ε there exists a positive real number $d(\varepsilon)$ such that if $v \in Y_s$ and $|v|_F<d(\varepsilon)$, then $|L_{x_n}(v)<\varepsilon/4$ for each n.

Since $|\ |_F$ is an F-norm, given $d>0$ there exists p such that $0<p<d$ and $|y|_F<p$ implies $|y/2|_F<d$. Moreover, since the metric space Y_s is compact, we can choose a p/2 net, $\{y_k\}_{k=1}^{t}$, for Y_s such that if $w \in Y_s$, there exists k, $1 \leqq k \leqq t$, with $|w-y_k|_F<p/2$. Furthermore, there exists an integer $N(\varepsilon)$ such that for $n>N$, $|L_{x_n}(y_k)|<\varepsilon/8$, $1 \leqq k \leqq t$.

Let x be an arbitrary element in Y_s. Then there exists a k, $1 \leqq k \leqq t$, such that $|x-y_k|_F <p/2$. Hence $|(x-y_k)/2|_F<d$. Since $(x-y_k)/2 \in Y_s$, for every $n>N$, we have

$$|L_{x_n}(x/2)| \leqq |L_{x_n}((x-y_k)/2)|+|L_{x_n}(y_k)|<\varepsilon/2.$$

Hence, altogether we have

$$|L_{x_n}(x)|<\varepsilon \text{ for each } n>N.$$

Since each x was an arbitrary element of Y_s, this implies that

$$|L_{x_n}|_{X^{**}}<\varepsilon \text{ for each } n>N.$$

Thus, $|x_n|_X<\varepsilon$ for each $n>N$, and this completes the proof of the theorem.

Before we go on further we cite some definitions due to Banach ([6], p. 243).

Let $\{X_k\}$ be a sequence of Banach spaces. We define

$$V(\{X_k\})=\left\{\{x_k\} : x_k\in X_k \text{ for each } k,\ \sum_{k=1}^{\infty}|x_k|_{X_k}<\infty\right\}.$$

If the vector addition is defined in the usual way, it is a linear set. Moreover, if we let

$$|x|_{V(X_k)}=\sum_{i=1}^{\infty}|x_i|_{X_i} \text{ and } x=\{x_i\},$$

then $(V(\{X_k\}), |\ |_{V(\{X_k\})})$ is a Banach space, and the dual of it is the space $(m(\{X_i^*\}), |\ |_{m(\{X_i^*\})})$, where

$$m(\{X_i^*\})=\left\{\{x_k\} : x_k\in X_k^* \text{ for each } k,\ \sup_k|x_k|_{X_k}<\infty\right\},$$

$$x_{m(\{X_k^*\})}=\sup_k|x_k|_{X_k^*},\quad x=\{x_k\}.$$

We define $(m_s(\{X_k^*\}), d)$ to be the Saks set $(m(\{X_k^*\}), |\ |_{m(\{X_k^*\})}, \|\ \|_s)$, where

$$\|\{x_k\}\|_s=\sum_{i=1}^{\infty}2^{-i}|x_i|_{X_i^*}/(1+|x_i|_{X_i^*}).$$

Then it is easy to see that $(m_s(\{X_k^*\}), d)$ satisfies (Σ).

Theorem. *The space* $(m_s(\{X_k^*\}), d)$ *is a complete metric space, thus it is a Saks space.*

Proof. Let $\{x_n\}$, $x_n=\{x_{n,k}\}$, be a Cauchy sequence of points from $m_s(\{X_k^*\})$. Then for each i, $|x_{n,i}-x_{m,i}|_{X_i^*}\longrightarrow 0$ as n, $m\longrightarrow\infty$.

Since X_i^* is complete, we can find $y_i\in X_i^*$ with $|y_i|_{X_i^*}<1$ such that $|x_{n,i}-y_i|_{X_i^*}\longrightarrow 0$ as $n\longrightarrow\infty$. Hence $y=\{y_i\}\in m_s(\{X_i^*\})$. Since coordinatewise convergence is equivalent to $\|\ \|_s$ convergence, we have $\|x_n-y\|_s\longrightarrow 0$ as n tends to infinity. Thus the given space is complete.

Theorem. *If each* X_i^* *is finite dimensional, then* $(m_s(\{X_i^*\}), d)$ *is compact.*

Proof. Let $\{X_n\}$, $x_n=\{x_{n,k}\}$, be a sequence of points from $m_s(\{X_k^*\})$. Then $|x_{n,k}|_{X_k^*}\leq 1$ for each n, k. X_k^* being a finite dimensional space, we can choose a subsequence $\{x_{n_k}\}$ of points from $m_s(\{X_k^*\})$ and a sequence $y=\{y_i\}$ such that $|x_{n_k,i}-y_i|_{X_i^*}\longrightarrow 0$ as $k\longrightarrow\infty$ for each i (Use Cantor diagonalization method.). Since coordinatewise convergence implies $\|\ \|_s$ convergence, we have $\|x_{n_k}-y\|_s\longrightarrow 0$ as $k\longrightarrow\infty$.

Since $(m_s(\{X_k^*\}), d)$ is complete by the above theorem, $y=\{y_k\}$ belongs to $m_s(\{X_k^*\})$. Thus $(m_s(\{X_k^*\}), d)$ is compact and the proof is completed.

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