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On the local structure of a Distribution

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1. Introduction

In the past, the properties of the figure, as a subject of Euclidean geometry, were considered in various ways, that its presentation in terms of the coordinates, was very limited. In Euclidean geometry based on the theory of manifold, in the broad sense, we study the length or area, and we also classify the properties of the figure which are invariant under the congruence transformations and the similarity transformations in R^n. In other words, according to the development of mathematics, the subjects of Euclidean geometry are gradually abstracted and classified as broad one.

In this sense, among many theories to study Euclidean geometry, there is the theory of distribution developed by Laurant Schwartz.

Let $x=(x_1, x_2, \cdots, x_n)$ and $y=(y_1, y_2, \cdots, y_n)$ be points of n dimensional Euclidean space R^n, and let $\alpha \in R$.

Then
$$x+y=(x_1+y_1, x_2+y_2, \cdots, x_n+y_n)$$
$$\alpha x=(\alpha x_1, \alpha x_2, \cdots, \alpha x_n)$$

Let $\varphi(x)$ be the infinite differentiable function on R^n. Then the closure $\overline{K}$ of $K=\{x;\ \varphi(x)\neq 0\}$ in R^n is called the support of $\varphi(x)$. When the support of $\varphi(x)$ is compact, the universal of $\varphi(x)$ is denoted by (D_{R^n}).

In that case, the mapping T which associates a complex number $T(\varphi)$ to each $\varphi(x)$ of (D_{R^n}) is called a distribution defined on (D_{R^n}) if the following two conditions ar satisfied
$$T(\alpha_1\varphi_1+\alpha_2\varphi_2)=\alpha_1 T(\varphi_1)+\alpha_2 T(\varphi_2) \qquad \text{(additivity)}$$
$$\lim_{\varphi_k\to 0} T(\varphi_k)=0 \qquad \text{(continuity)}$$

Thus a distribution T is a continuous additive functional defined on (D_{R^n})

The subject of this paper is to study and extend the theory of distribution of Schwartz. This work is so broad that this paper focused on the local structure of the distribution.

This field has been presented only by Kim H. B. in Dan Kook university at the meting of the Korean Mathematical Society in 1969. Thus this study field needs much more research.

This paper is divided into the following three parts

1. Local structure of the distribution

2. The support of a distribution

3. Distribution whose support is compact

At the end of this paper we will give a necessary and sufficient condition for a distribution T to be have x=0 as its support.

2. Local structure of the distribution

Lemma 1. Let K be a compact set of R^n and $\varphi(x)\in(D_{R^n})$, where sup (φ) is a subset of K. Let $\mathrm{dis}_K(\varphi, \Psi)$ be the distance between two functions φ and Ψ in (D_K). For each $\varphi\in(D_K)$, let $\rho_K^{(m)}(\varphi)=\sup_{|p|\leq m, x\in K}|D^{(p)}\varphi(x)|$, i.e., $\rho_K^{(m)}$ is the maximum value in K of absolute value of all the partial differentials lower then m order. Therefore, if $m<p$ then $\rho_K^{(m)}(\varphi)\leqq\rho_K^{(p)}(\varphi)$.

Thus the next conditions are satisfiea

i) $\mathrm{dis}_K(\varphi, \Psi)=\mathrm{dis}_K(\varphi-\Psi, 0)=\mathrm{dis}_K(\phi-\Psi, 0)=\mathrm{dis}_K(\phi, \Psi)$

ii) $\mathrm{dis}_K(\varphi, \Psi)\geqq 0$, and $\mathrm{dis}_K(\varphi, \Psi)=0$ if and only if $\varphi=\Psi$

iii) $\mathrm{dis}_K(\varphi, \Psi)\leqq\mathrm{dis}_K(\varphi, x)+\mathrm{dis}_K(x, \Psi)$

iv) $\varphi_h\to 0$ $((D_K))$ if and only if $\lim_{h\to\infty}\mathrm{dis}_K(\varphi, 0)=0$

Lemma 2. The metric space (D_K) is complete. This is. if a sequence $\{\varphi_h\}$ of functions in (D_K) satisfies Cauchy condition $\lim_{h,p\to\infty}\mathrm{dis}_K(\varphi_h, \varphi_p)=0$ then there exists only one $\varphi\in(D_K)$ such that

$$\lim_{h\to\infty}\mathrm{dis}_K(\varphi_h, \varphi)=0$$

Lemma 3. Let T be a functional defined on (D_{R^n}) such that

$$T(\alpha_1\varphi_1+\alpha_2\varphi_2)=\alpha_1 T(\varphi_1)+\alpha_2 T(\varphi_2) \qquad \text{(additivity)}$$

Then T is a distribution if and only if

$$\lim_{h\to\infty}\mathrm{dis}_K(\varphi_h, 0)=0 \text{ implies } \lim_{h\to\infty}T(\varphi_h)=0$$

for any compact set $K\subset R^n$.

Definition. The order of T at K is said to be at most m, if

$$\lim_{h\to\infty}\rho_K^{(m)}(\varphi_h)=0 \text{ implies } \lim_{h\to\infty}T(\varphi_h)=0$$

Now, the minimum value of m is called the order of T at K. Let T be an additive functional defined on (D_{R^n}). If there exists an integer $m\geqq 0$ such that

$$\lim_{h\to\infty}\rho_K^{(m)}(\varphi_h)=0 \text{ implies } \lim_{h\to\infty}T(\varphi_h)=0$$

for any compact set $K\subset R^n$, then T is a continuous distribution on (D_{R^n}).

In this case, the order of T in R^n is at most m, and minimum value of m is called the order in R^n.

Lemma 4. The order of a distribution T on any compact set $K\subset R^n$ is finite.

Proof The distribution T is continuous on the metric space (D_K) by lemma 3.

Since $T(0)=0$, there exists a positive number $\delta=\delta(T, K)$ such that

$$\varphi\in(D_K) \text{ and } \mathrm{dis}_K(\varphi, 0)\leqq\delta \text{ implies } |T(\varphi)|\leqq 1 \quad \cdots\cdots(1)$$

Here, $\delta(T, K)$ means that δ depends on T and K. By the way, for every $m_0\geqq 0$

$$\mathrm{dis}_K(\varphi,0)=\sum_{m=0}^{m_0}\left(\frac{1}{2^m}\ \frac{\rho_K^{(m)}(\varphi)}{1+\rho_K^{(m)}(\varphi)}\right)+\sum_{m=m_0+1}^{\infty}\left(\frac{1}{2^m}\ \frac{\rho_K^{(m)}(\varphi)}{1+\rho_K^{(m)}(\varphi)}\right)$$

Using this fact we obtain, if m<p, then $\rho_K^{(m)}(\varphi)\leqq\rho_K^{(p)}(\varphi)$,

$$\mathrm{dis}_K(\varphi,0)\leqq\frac{\rho_K^{(m)}(\varphi)}{1+\rho_K^{(m_0)}(\varphi)}\cdot\sum_{m=0}^{m_0}2^{-m}+\sum_{m=m_0+1}^{\infty}2^{-m}$$

It follows from (1) that there exist an integer $m_0=m_0(\mathrm{T},\ K)\geqq0$ and a positive number $\eta=\eta(\mathrm{T},\ \mathrm{K},\ \mathrm{m}_0)$ such that

$$\varphi\in(D_K)\ \text{and}\ \sup_{|p|\leqq m_0,\,x\in K}|D^{(p)}\varphi(x)|\leqq\eta\ \ \text{implies}\ \ |T(\varphi)|\leqq1 \quad\cdots\cdots(2)$$

Next, we introduce a symbol

$$\frac{\partial^s\varphi}{\partial x^s}=\frac{\partial^{ns}\varphi}{\partial x_1^s\partial x_2^s\cdots\partial x_n^s}\quad\cdots\cdots(3)$$

Then we can show that there exists a positive number $\kappa=\kappa\ (\eta, m_0)$ such that

$$\varphi\in(D_K)\quad\text{and}\ \int_K\left|\frac{\partial^{m_0+1}\varphi(x)}{\partial x^{m_0+1}}\right|^2dx\leqq\kappa\ \ \text{implies}\ \sup_{|p|\leqq m_0,\,x\in K}|D^{(p)}\varphi(x)|\leqq\eta\quad\cdots\cdots(4)$$

Let t be the largest one among 1 and the diameter of a compact set K.

By Schwartz inequality

$$\int_K\left|\frac{\partial^{m_0+1}\varphi(x)}{\partial x^{m_0+1}}\right|dx\leqq\left(\int_K dx\cdot\int_K\left|\frac{\partial^{m_0+1}\varphi(x)}{\partial x^{m_0+1}}\right|^2dx\right)^{\frac{1}{2}}$$

hence, if φ satisfies (4) then

$$\int_K\left|\frac{\partial^{m_0+1}\varphi(x)}{\partial x^{m_0+1}}\right|dx\leqq t^{\frac{n}{2}}\cdot\kappa^{\frac{1}{2}}\quad\cdots\cdots(5)$$

On the other hand, if the support of $\Psi(x)$ is compact then

$$\phi(x_1,,x_n)=\int_{-\infty}^{x_i}\frac{\partial\Psi(x_1,\cdots,x_{i-1},t,x_{i+1},\cdots,x_n)}{\partial t}\,dt,\quad\text{for } i=1,2,\cdots,n$$

Therfore

$$|\phi(x_1,\cdots,x_n)|\leqq\int_{-\infty}^{x_i}\left|\frac{\partial\Psi(x_1,\cdots,x_{i-1},t,x_{i+1},\cdots,x_n)}{\partial t}\right|dt\quad\cdots\cdots(6)$$

$$\text{and}\quad\left|\frac{\partial^{m_0}\varphi(x)}{\partial x^{m_0}}\right|\leqq\int_K\cdots\int_K\left|\frac{\partial^{m_0+1}\varphi(x)}{\partial x^{m_0+1}}\right|dx_1,\cdots,dx_n\ .$$

Thus, by (5), if φ satisfies the condition (4) then

$$\left|\frac{\partial^{m_0}\varphi(x)}{\partial x^{m_0}}\right|\leqq t^{\frac{n}{2}}\cdot\kappa^{\frac{1}{2}}\quad\cdots\cdots(7)$$

Hence, Using (6) again for t≧1, we have

$$\sup_{|p|\leqq m_0,\,x\in K}|D^{(p)}\varphi(x)|\leqq t^{\frac{n}{2}}\cdot\kappa^{\frac{1}{2}}\cdot t^{nm_0}\quad\cdots\cdots(8)$$

Now if we take κ

$$\kappa\leqq\eta^2t^{-2nm_0-n}\quad\cdots\cdots(9)$$

then

$$\sup_{|p|\leqq m_0,\,x\in K}|D^{(p)}\varphi(x)|\leqq\eta$$

Finally, we can find the following:

If a distribution T and a compact set K are given, then there exist an integer $m_0=m_0(\mathrm{T},\ K)\geqq0$ and a positive number $\kappa=\kappa\ (\mathrm{T},\ K, m_0)$ such that

$$\varphi\in(D_K),\ \int_K\left|\frac{\partial^{m_0+1}\varphi(x)}{\partial x^{m_0+1}}\right|^2dx\leqq\kappa\ \ \text{implies}\ \ |T(\varphi)|\leqq1\quad\cdots\cdots(10)$$

For each $\varphi(x)\in(D_K)$, define $\Psi=L(\varphi)$ by

$$\Psi(x)=\frac{\partial^{m_0+1}\varphi(x)}{\partial x^{m_0+1}}\in(D_K)$$

Then L is additive, so $L(\alpha_1\varphi_1+\alpha_2\varphi_2)=\alpha_1L(\varphi_1)+\alpha_2L(\varphi_2)$, and it is one-to-one by (6)

Let $$S(\Psi)=T(L^{-1}(\Psi))=T(\varphi) \quad\cdots\cdots(11)$$

Then S is an additive functional on the vector space

$$M=\left\{\Psi:\Psi=L\Psi=\frac{\partial^{m_0+1}\varphi(x)}{\partial x^{m_0+1}},\ \varphi\in(D_K)\right\}$$

By (10) $$\|\Psi\|_K=\left(\int_K|\Psi(x)|^2dx\right)^{\frac{1}{2}}\leqq\kappa^{\frac{1}{2}} \text{ implies } |S(\Psi)|\leqq1 \quad\cdots\cdots(12)$$

The vector space M is a pre-Hilbert space equipped with the norm $\|\Psi\|$.

Therefore, from the theorem of Riesz it follow that there exists a complex-value measurable function $f(x)=f(x;\ T,\ K)$ defined on K with $\|f\|_K\leqq\kappa^{\frac{1}{2}}$ such that

$$S(\Psi)=\int_K f(x)\Psi(x)dx \quad\cdots\cdots(13)$$

From the above facts, the next lemma holds, and the order of K is at most $n(m_0+1)$ in K.

Lemma 5. For every distribution T and compact set $K\subset R^n$, for some integer $m_0=m_0(T,\ K)\geqq0$ and $\kappa=\kappa(T,\ K,\ m_0)>0$, there exists $f(x)=f(x;\ T,\ m_0)$ such that

$$\|f\|_K=\left(\int_K|f(x)|^2dx\right)^{\frac{1}{2}}\leqq\kappa^{\frac{1}{2}} \text{ and } T(\varphi)=\int_K f(x)\frac{\partial^{m_0+1}\varphi(x)}{\partial x^{m_0+1}}dx, \text{ for every } \varphi\in(D_K)$$

Lemma 6. Let $\{U_\lambda;\ \lambda\in\Lambda\}$ be an open covering of R^n, that is every U_λ is open in R^n and $\bigcup_{\lambda\in\Lambda}U_\lambda=R^n$. Then there exists a countable family $\{V_i\}$ of open sets which satisfies the following conditions

i) Every V_i is contained in any U_λ.

ii) $\bigcup_{i=1}^{\infty}V_i=R^n$.

iii) $\{V_i\}$ is locally finite, that is for every compact set $K\subset R^n$, V_i is finite which $V_i\cap K$.

iv) The closure of each V_i is compact.

Lemma 7. Let $\{U_\lambda;\ \lambda\in\Lambda\}$ be an open covering of R^n. Then there exists a countable family $\{\varphi_i(x)\}\subset(D_{R^n})$ which satisfies the following conditions.

i) The support of each $\varphi_i(x)$ is contained in any U_λ.

ii) $0\leqq\varphi_i(x)\leqq1$ $(i=1,2,\cdots)$, and $\sum_{i=1}^{\infty}\varphi_i(x)=1$.

iii) {supp (φ_i)} is locally finite. (supp (φ_i) is support of φ_i)

3. The support of a distribution

We say that a distribution T is 0 at the neighborhood of point x^0, if there exists neighborhood $U(x^0)$ of x^0 such that

$$\text{supp}\ (\varphi)\subset U(x^0) \text{ implies } T(\varphi)=0$$

Lemma 8. If a distribution T is 0 at the neighborhood of each point in an open set

$U \subset R^n$, then T is 0 at U, that is $T(\varphi)=0$ for all $\varphi \in (D \cdot)$ with supp $(\varphi) \subset U$.

(Proof) For each $x^0 \in U$, we will take neighborhood $U(x^0)$ of x^0 contained in U such that

supp $(\varphi) \subset (x^0)$ implies $T(\varphi)$

Then $\{U(x^0);\ x^0 \in U\}$ is an open covering of U. Hence, by lemma 7, there exists a countable family $\{\varphi_i(x)\} \subset (D_{R^n})$ which satisfies the following conditions

i) The support of each $\varphi_i(x)$ is contained in $U(x^0)$ $\cdot$

ii) $0 \leqq \varphi_i(x) \leqq 1$ $(i=1, 2, \cdots)$, and $\sum_{i=1}^{\infty} \varphi_i(x)=1$.

iii) {supp (φ_i)} is locally finite in U.

Now, if supp$(\varphi) \subset U$ then, by ii), $T(\varphi)=T((\sum_{i=1}^{\infty} \varphi_i(x) \cdot \varphi(x))$ and, by iii), $\sum_{i=1}^{\infty} \varphi_i(x) \cdot \varphi(x)$ is a finite sum on every compact subset of U.

Therefore, $T(\varphi)=\sum_{i=1}^{\infty} T(\varphi_i \cdot \varphi)$ by the additivity of T. Since, supp$(\varphi_i\ \varphi) \subset$supp (φ_i) is contained in any $U(x^0)$ by i), from the definition of $U(x^0)$ it follows that $T(\varphi_i\ \varphi)=0$

Therefore, if supp(φ) $\subset U$ then $T(\varphi)=0$.

4. Distribution whose support is compact

Definition of (D_{R^n}) : The universal set of infinite partially differentiable functions $\varphi(x)$ defined on R^n is a vector space with respect to operations

$$(\varphi+\Psi)(x)=\varphi(x)+\Psi(x),\quad (\alpha\varphi)x=\alpha\varphi(x)$$

A series $\{\varphi_h(x)\}$ of functions in (D_{R^n}) is said to be convergent to 0 if for any $\varphi_h(x)$, its all palial derivative functions $\frac{\partial \varphi_h(x)}{\partial x_i}$, $\frac{\partial^2 \varphi_h(x)}{\partial x_i \partial x_j}$, $\cdots$ is uniformly convergent to 0 on every compact set K. We will denote it by

$$\varphi_h \Rightarrow 0\ ((D_{R^n}))\ \text{or}\ \lim_{h\to\infty} \varphi_h=0\ ((D_{R^n}))$$

Lemma 9. (D_{R^n}) equipped with

$$\text{dis}_E(\varphi,\ \Psi)=\sum_{k=1}^{\infty} \frac{1}{2^k}\ \frac{\text{dis}_{E,k}(\varphi, \Psi)}{1+\text{dis}_{E,k}(\varphi, \Psi)},$$

where $$\text{dis}_{E,k}(\varphi, \Psi)=\sum_{m=0}^{\infty} \frac{1}{2^m}\ \frac{\rho_k^{(m)}(\varphi-\Psi)}{1+\rho_k^{(m)}(\varphi-\Psi)}$$

is a complete metric space.

And $\varphi_h \Rightarrow 0$ $((D_{R^n}))$ if and only if $\lim_{h\to\infty}$ dis$_E$ $(\varphi_h,\ 0)$

Definition of $(D_{R^n})'$, $(E_{R^n})'$: A distribution T is an additive continuous functional. The universal set of a distribution in the dual space of (D_{R^n}), which is denoted by $(D_{R^n})'$.

Let T be an additive continuous functional such that

$$\lim_{h\to\infty} \varphi_k=0\ ((E_{R^n}))\ \text{implies}\ \lim_{h\to\infty} T(\varphi_h)=0$$

Let (E_{R^n}) be the dual space of (E_{R^n}). Then we obtain the following lemma.

Lemma 10. The univessal set of a distribution whose support is equal to $(E_{R^n})'$. By (10), for open set $U_i \supset$(clesure of U)$\supset$supp (T) which contains compact closure K_1, if we take (D_{K_1}) then, $\alpha\varphi \in (D_{K_1})$ for every $\varphi \in (E_{R^n})$

Hence, by lemma 5, for some integer $p=p(T,\ K_1) \geqq 0$ and $\kappa=\kappa(T, k_1, p)>0$, there exists

$f(x)=f(x;\ T,\cdot k_1,\ p)$ such that

supp (f)$\subset$K, $\|f\|_{K_1}=\int_{K_1}|(fx)|^2dx^{\frac{1}{2}}\leqq\kappa^{\frac{1}{2}}$ and $F(\varphi)=T(\alpha\varphi)=\int_{K_1}f(x)(D^{(p)}\cdot\alpha\varphi)(x)dx$

Then, by the Leibnitz fomular the following equality holds for $D^{(p)}\alpha(x)$

$$F(\varphi)=T(\alpha\varphi)=\sum_{q\leqq p}C_q^p\int f(x)D^{(p-q)}\alpha(x)\cdot D^{(p)}\varphi(x)dx$$
$$=\sum_{q\leqq p}(-1)^{|q|}D^{(q)}T_{f\cdot D^{(p-q)}\alpha}(\varphi)$$

Since $\mathrm{supp}(f\cdot D^{(p-q)}\alpha)\subset\mathrm{supp}(f)\subset K$, we obtain the following lemma.

Lemma 11. If the support of a distribution T is compact, then T is sepresented by

$$T=\sum_{q\leqq p}D^{(q)}T_{f\cdot g(x)}.$$

Here, we may assume that the support of each function $f_q(x)$ contains every nighborhood of the support of T. Clearly, the order of T in R^n is finite ($\leqq|p|$). Using the formula of lemma 11, we oftain the next main theorem.

Theorem. Distribution T has x=0 as its support if and only if T is represented by

$$T=\sum_{|q|\leqq|p|}C_qD^{(p)}T_s \quad \cdots\cdots(1)$$

where the C_q are constant.

Proof By lemma 11, the order of T is finite. Hence, for a given function series $\{\varphi_h(x)\}$ of (D_{R^n}), if $\{D^q\varphi_h(x)\}$, $|q|\leqq|p|$, is uniformly convergent to 0 in R^n then $\lim_{h\to\infty}T(\varphi_h)=0$.

For every $\Psi(x)\in(D^{R^n})$, let

$$\Psi(x)=\sum_{|q|\leqq|p|}\frac{x_1^{q_1}\cdots x_n^{q_n}}{q_1!\cdots q_n!}(D^{(q)}\Psi(x))_{x=0}+\varphi(x)$$

which is the Taylor extension of $\Psi(x)$.

Then

$$(D^{(p)}\varphi(x))_{x=0}=0 \quad (|q|\leqq|p|) \quad \cdots\cdots(2)$$

At first, we will derive a formula for $T(\varphi)$ from (2).

(i) From (2), it follows that $\mathrm{Sup}_{|q|\leqq|p|,\,|x|\leqq\frac{1}{h}}|D^{(q)}\varphi(x)|=\eta(h)$ is convergent to 0 ($h\to\infty$)

Hence, since $\dfrac{\partial^{p_1-1+p_2+\cdots+p_n}\varphi(x)}{\partial x_1^{p_1-1}\partial x_2^{p_2}\cdots\partial x_n^{p_n}}=\int_0^{x_1}D^{(p)}\varphi(t,x_2,\ldots,x_n)dt$

which is made of $(D^{(p)}\varphi(x))_{x=0}=0$, we obtain

$$\sup_{|x|\leqq\frac{1}{h}}\left|\frac{\partial^{p_1-1+p_2+\cdots+p_n}\varphi(x)}{\partial x_1^{p_1-1}\partial x_2^{p_2}\cdots\partial x_n^{p_n}}\right|\leqq\frac{1}{h}\eta(h).$$

Again by (2), if $|m|\leqq|p|$ then

$$\sup_{|x|\leqq\frac{1}{h}}|D^{(m)}\varphi(x)|\leqq\eta(h)\left(\frac{1}{h}\right)^{|p|-|m|} \quad \cdots\cdots(3)$$

(ii) If $\alpha(x)\in(D_{R^n})$ is defined by $\alpha(x)=\begin{cases}1, & |x|\leqq\frac{1}{2}\\ 0, & |x|\geqq1\end{cases}$ $\cdots\cdots(4)$

then

$$D^{(g)}\alpha_h(x)=h^{|q|}(D^{(g)}\alpha(h))_{y=hx}, \text{ for } \alpha_h(x)=\alpha(hx)$$

Hence, for some positive integer e_q

$$\sup_{|x|\leq\frac{1}{h}} |D^{(q)}\alpha_h(x)| = \sup_{x\in R^n} |D^{(q)}\alpha_h(x)| \leq h^{|q|}e_q \cdots\cdots\cdots\cdots (5)$$

Now, set $\varphi_h(x)=\alpha_h(x)\ \varphi(x)$, then by the Leibnitz formula,

$$D^{(r)}\varphi(x)=\sum_{q\leq r} C_q^r D^{(r-q)}\alpha_h(x)\cdot D^{(q)}\varphi(x)$$

Therefore, since $\text{supp}(\varphi_h)\subset\text{supp}(\alpha_h)\subset\left\{x;|x|\leq\frac{1}{h}\right\}$,

from (3) and (5) it follow that

$$\sup_{x\in R^n}|D^{(r)}\varphi_h(x)| = \sup_{|x|\leq\frac{1}{h}} |D^{(r)}\varphi_h(x)| \leq \sum_{q\leq r} C_q^r e_{r-q} h^{|r-q|}\cdot|h|^{-|p|+|q|}\eta(h)$$

Since $r\geqq q$, $|r-q|=|r|-|q|$, and $\eta(h)\to 0$ as $h\to\infty$, we obtain

$$\sup_{x\in R^n}|D^{(r)}\varphi_h(x)|\to 0\ (h\to\infty),\ |r|\leq|p|$$

Thus $\lim_{h\to\infty} T\ (\varphi_h(x))=0$, that is, $\lim_{h\to\infty} T(\alpha(hx)\varphi(x))=0$

Finally, since $T(\varphi)=0$ it follows from (2) that

$$T(\Psi)=\sum_{|q|\leq|p|} T\left(\frac{x_1^{q_1}\cdots x_n^{q_n}}{q_1!\cdots q_n!}\right)\cdot(D^{(q)}\Psi(x))_{x=0}$$

Therefore

$$T=\sum_{|q|\leq|p|} C_q D^{(q)} T_\delta$$

Corollary. Define T_{δ_h} by $T_{\delta_h}(\varphi)=\varphi(h)$. Then a distribution T has only one point $x=h$ as its support if and only if T is represented by

$$T=\sum_{|q|\leq p} C_q D^{(q)} T_{\delta_h}$$

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