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DP Property, RNP and Semiseparability

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1. Introduction

A Banach space X is said to have *DP* (weak* *DP*) [*DP*≡Dunford-Pettis] property if for every weakly convergent to zero sequence $\{x_n\}$ in X and for every weakly (weak*) convergent to zero sequence $\{x_n^*\}$ in X^*, $\lim_{n\to\infty} x_n^* x_n=0$. Rosenthal [11] and Pelczynski [6] obtain interesting results for a Banach space X that has *DP* property with X^* *WCG* (weakly compactly generated). It is well known that conjugate *WCG* spaces have *RNP* (Radon-Nikodym property). In section two we extend the results of Rosenthal [11] and Pelczynski [6] to the conjugate Banach spaces having *RNP*. In the same vein we also generalize some results of Howard and Melendez [4] that are valid for conjugate *WCG* spaces to conjugate Banach spaces having *RNP*; and it has been accomplished in the latter part of section two.

Restrepo [10] proved that, "*A separable Banach space X has an equivalent Frechet differentiable norm if and only if X^* is separable*". The concept of Frechet differentiability has been extended to very smoothness in one direction, and to property (I) in another direction. In section three, we show that the result of Restrepo continues to hold if Frechet differentiability is replaced by very smoothness; it is also known to be true for property [I] [8]. It is now natural to ask: Can Restrepo's result be generalized by replacing separability of X by some weaker concept? In section three we show that the above result cannot be extended if separability is replaced by semiseparability, a concept to be found in Howard and Melendez [4]. We show this by reproducing a counterexample which is due to Johnson and Lindenstrauss [5]. In the course of our investigation we have also sharpened a result of Whitfield [12] in the form: A Banach space that admits an equivalent rough norm cannot admit an equivalent very smooth norm.

In the final section, we observe that there is no connection between the *DP* property, semiseparability and property (I) of Mazur.

The discussion of some of the relevant concepts has been postponed to the respective sections.

2. We first prove the following Result that generalises the Result of H. P. Rosenhtal [11, Theorem 2.1].

RESULT 2.1. *Let the Banach space X satisfy DP. Then if X is isomorphic to a subspace of a conjugate Banach space that satisfies RNP then every weak cauchy sequence in X converges in the norm topology of X.*

PROOF. As observed by Rosenthal [11], since X is assumed to satisfy *DP*, given (x_n) and (f_n) sequences in X and X^* respectively such that $x_n \to 0$ weakly, and (f_n) is weak cauchy, $f_n(x_n) \to 0$.

Now we note that it is enough to show that every sequence which is weakly convergent to zero in X converges in the norm topology of X. This follows immediately from the fact that a sequence (y_n) in a Banach space Y is a (weak) cauchy sequence if and only if for every two increasing sequences of indices (p_m) and (q_m) the sequence $(y_{p_m} - y_{q_m})$ is weakly convergent to zero.

Now since the property *DP* is linear topological, we may suppose that there is a Banach space B with B^* satisfies *RNP* and $X \subset B^*$. Let (x_n) in X with $x_n \to 0$ weakly; we shall show that $\|x_n\| \to 0$. Suppose not; we may assume, by passing to a subsequence if necessary, that there is a $\delta > 0$ with $\|x_n\| \geq \delta$ for all n.

Now choose b_n in B with $\|b_n\| = 1$ and $|x_n(b_n)| > \delta$ for all n. Since B^* satisfy *RNP*, we can extract a weak-cauchy subsequence (b_{n_i}) from (b_n)(cf. Diestel [1, p. 246]). Now define $T: B \to X^*$ by $(Tb)(x) = x(b)$ for all b in B and x in X. T is a continuous linear operator, and so (Tb_{n_i}) is a weak cauchy sequence in X^*. Thus by the first observation, $\lim_{i\to\infty} (Tb_{n_i})(x_{n_i}) = 0 = \lim_{i\to\infty} x_{n_i}(b_{n_i})$ a contradiction. This completes the proof of the Result.

We can also prove the following Result, the proof of which, we shall omit as it runs exactly on the lines of Pelczynski [6].

RESULT 2.2. *Lei X be a Banach space with X^* satisfing RNP. Let X satisfy property DP. Then every weak cauchy sequence in X^* converges in the norm topology of X^*.*

In [8] the authors have proved the following result, in which they have used the same technique as used above.

RESULT. *If X has DP and is isomorphic to a subspace of Y^* where $Y \not\supset l^1$*

then X has Schur property (i.e. every weak cauchy sequence in X is norm convergent).

In fact, the following elegant characterisation for the dual spaces having Schur property has been established.

RESULT. *X^* has the Schur property if and only if X has DP and $X \not\supset l^1$.*

This result can be used to prove the following rather surprising result which was pointed out to us by Professor A.L. Brown of the University of New Castle upon Tyne and our thanks are due to him.

RESULT. *If $C(K)$ is separable then $C(K) \not\supset l^1$ if and only if $C(K)^* = l^1$.*

PROOF. Since $C(K)$ has the *DP* property, the above result implies that $C(K)^*$ has the Schur property. Noting that K is metrizable, we use Milutin's result (viz.: Let K be a compact metric uncountable space. Then $C(K)$ is isomorphic to $C(0, 1)$) to conclude that the above statement equivalently means that K is countable as $C[0, 1]$ is universal for all separable spaces. Thus we see that K is dispersed and hence every measure on K is generated by its atoms implying that $C(K)^*$ is, in fact, isometric to l^1.

We shall now give some applications of the Results 2.1 and 2.2.

RESULT 2.3. *Let X^* satisfies DP. Then X^* is WCG if and only if X^* is separable.*

PROOF. Let X^* be *WCG*. Let K be weakly compact subset of X^* that generates X^*. By Eberlein's Theorem K is weakly sequentially compact; that is from any sequence (x_n) in K we can extract a weakly convergent subsequence which is obviously weak cauchy and hence norm convergent. So K is sequentially compact and hence compact. Hence X^* is generated by a compact set and hence is separable.

The following result can also be proved on the lines of the above proof.

RESULT 2.4. *If X^* is DP and satisfies RNP then every weakly compact set in X^* is compact and hence separable.*

We say that a Banach space is *almost reflexive* if every bounded sequence in X contains a weak cauchy subsequence.

The next two results generalise Theorem 9 and Corollary 10 of [4].

RESULT 2.5. *If X is a conjugate Banach space that satisfies RNP, the X* *has no infinite dimensional subspace isomorphic to an almost reflexive Banach space with the DP property. In particular, X has no subspace isomorphic to* C_0.

PROOF. Let Y be an almost reflexive Banach space with the DP property. If Y is ismorphic to a subspace of X, then by Result 2.1, every weak cauchy sequence in Y converges in the norm topology on Y. If $\{y_n\}$ is a sequence in the unit disk of Y, then $\{y_n\}$ has a weak cauchy subsequence $\{y_{n_i}\}$ since Y is almost reflexive. Thus $\{y_{n_i}\}$ coverges in the norm topology and hence the unit disk of Y is compat. Hence Y is finite dimensional; a contradiction.

RESULT 2.6. *If* X^* *satisfies DP and RNP then* X^{**} *cannot be WCG.*

PROOF. We just note that the hypothesis implies that X has DP property and is almost reflexive. Applying result 2.5 we get the desired conclusion.

In fact, more generally we have the following

RESULT 2.7. *If* X^* *satisfies DP and RNP then* X^{**} *cannot even satisfy RNP.*

3. We recall the definition of semiseparability: A Banach space is said to be *semiseparable* if its conjugate X^* contains a sequence (f_n) that is total over X.

We now observe

RESULT 3.1. *In a semiseparable space X, every weakly compact set is separable.*

PROOF. If K is weakly compact set in X, then K is metrizable in the weak topology [13] (More general result can be found in [3, p. 71]) and so K is weakly separable and hence separable.

In [5] the example of a Banach space U has been constructed to satisfy the following properties.

a) U^* is W^*-separable and U is not WCG

b) U^* is WCG with an unconditional basis

c) U has an equivalent Frechet differentible norm.

The authors already constructed there a countable subset of U^* that is total over U and hence the space is semiseparable and so U^* is W^*-separable, It is, in fact, because of Result 3.1, the space ceases to be WCG.

Thus here is an example of a semiseparable Banach space that admits an equivalent Frechet differentiable norm, the dual of which is not separable.

4. Whitfield [12] showed that Banach space that admits rough norm cannot ad-

mit an equivalent Frechet differentiable norm. We now sharpen this result in the following.

RESULT 4.1. *If X admits an equivalent rough norm then X cannot admit an equivalent very smooth norm.*

PROOF. If X admits an equivalent rough norm then there is a separable subspace Y of X such that Y^* is nonseparable. By a well-known characterization this in turn implies that X^* cannot possess *RNP* and hence X cannot admit an equivalent very smooth norm.

We note that this result cannot be improved further as the space l^1 has a rough norm as well as it admits an equivalent smooth norm.

As an application of the above result we now have the stipulated generalization of Restrepo's theorem.

RESULT 4.2. *A separable Banach space X admits an equivalent very smooth norm if and only if X^* is separable.*

1ST PROOF. 'If' part follows by Restrepo's theorem. 'Only if' part: If X has an equivalent very smooth norm then X^* possesses *RNP* and so every separable subspace has a separable dual and so choosing in particular X itself as a separable subspace we conclude that X^* is separable.

2ND PROOF. If X admits an equivalent very smooth norm then by the above result X cannot admit an equivalent rough norm and so the density characters of X and X^* are the same, implying that X^* is separable.

We remark here as a consequence that if on a separable space there exists an equivalent very smooth norm then there does exist an equivalent Frechet differentiable norm and so a separable space need not possess an equivalent very smooth norm (eg. l^1, $C[a, b]$).

In this connetion, we mention that we know of no example of a Banach space with very smooth norm that does not admit an equivalent Frechet differentiable norm.

5. We conclude the paper by observing that there is no interconnection between these properties. More explicitly we have

(a) As noted in [4] there is no relationship between W^*DP and semiseparability. C_0, $C[0,1]$ are semiseparable [even separable] but they do not possess W^*DP, More generally l_p>$(1<p<\infty)$ spaces are separable and do not satisfy *DP* property as they are infinite dimensional reflexive spaces.

On the other hand, let X be a *WCG* Banach space which is not separeble e.g. $C_0(\Gamma)$, Γ uncountale). Let K be a weakly compact fundamental set in X. Then the space $C(K)$ cannot be semiseparable but satisfies *DP* property.

(b) There is no relation between property (I) and *DP* property. Phelps [9] has renormed the space l^1 in such a way that it lacks property (I). We observe here that since property *DP* is linear topological invariant and *W* DP* impies *DP*; the space l^1 in this norm satisfies *DP*.

On the other hand since any reflexive space can be renormed to have a Frechet differentiable norm, by the Result of Pethe and Thakare [7], the space with the proper norm satisfies property (I). If we take the space to be infinite dimensinal, the space lacks *DP* property.

(c) It is even more easy to observe that there is no relation between property (I) and semiseparability (even separability).

Examples: 1. l^1

2. Any reflexive nonseparable space (Note that any reflexive nonseparable space is automatically nonsemiseparable).

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