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INTEGRAL REPRESENTATION OF APPELL'S FUNCTIONS

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1. Introduction

The object of this paper is to obtain integral representations of Appell's functions [3] with the help of fractional integration and differentiation.

For the sake of convenience the rules of fractional integration and differentiation are given below. Following Erdely; [2], we write the rules of fractional integration by parts in the form

$$\int_a^b u\frac{d^w v}{d(b-x)^w}dx=\int_a^b v\frac{d^w u}{d(x-a)^w}dx \tag{1}$$

The fractional derivatives in [1] can be defined by integrals, if the real part of w is negative. Thus

$$\frac{d^w u}{d(x-a)^w}=\frac{1}{\Gamma(-w)}\int_a^x (x-y)^{-w-1}\ u(y)\ dy \tag{2}$$

$$\frac{d^w v}{d(b-x)^w}=\frac{1}{\Gamma(-w)}\int_x^b (y-x)^{-w-1}\ v(y)\ dy \tag{3}$$

where $R(w)<0$.

If u and v are expressible by means the series

$$\begin{aligned} u&=\sum_{r=0}^{\infty} C_r (x-a)^{\lambda+r-1}\\ v&=\sum_{s=0}^{\infty} A_s (b-x)^{\rho+s-1}\end{aligned} \tag{4}$$

then the fractional derivatives are obtained by differentiating these series term by term and using the definiton.

$$\frac{d^w v^{\mu-1}}{dv^w}=\frac{\Gamma(\mu)}{\Gamma(\mu-w)}\ v^{\mu-w-1} \tag{5}$$

for the fractional derivative, which holds for all values of w except $w=\mu$.

In our investigation we require the result

$$\frac{d^w}{d(1-x)^w}[x^{-\delta}(1-x)^{\lambda-1}]=\frac{\Gamma(\lambda)}{\Gamma(\lambda-w)}(1-x)^{\lambda-w-1}2F_1(\delta,\lambda;\lambda-w;1-x) \tag{6}$$

and the definition of J. Kampe de Feriets' hypergeometric series of two variables [1].

$$F\left(\begin{array}{c|l|c} l & a_1, a_2, \cdots, a_l & \\ m & b_1, b_2, \cdots, b_m; b'_1, b'_2, \cdots, b'_m & x, y \\ n & c_1, c_2, \cdots, c_n & \\ p & d_1, d_2, \cdots, d_p; d'_1, d'_2, \cdots, d'_p & \end{array}\right)$$

$$= \sum_{p_1=0}^{\infty} \sum_{q_1=0}^{\infty} \frac{\prod_{i=1}^{l}(a_j)_{p_1+q_1} \prod_{h=1}^{m}(b_h)_{p_1}(b'_h)_{q_1}}{\prod_{j=1}^{n}(c_j)_{p_1+q_1} \prod_{h=1}^{p}(d_h)_{p_1}(d'_h)_{q_1}} \frac{x^{p_1}}{p_1!} \frac{y^{q_1}}{q_1!} \tag{7}$$

2. The following results are to be established in this section

$$\int_0^1\int_0^1 u^{b_1-1} v^{b_2-1} (1-u)^{c_1-b_1-1} (1-v)^{c_2-b_2-1} F_1(a, c_1, c_2; d; ux, vy) du\, dv$$

$$= \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(c_1-b_1)\Gamma(c_2-b_2)}{\Gamma(c_1)\Gamma(c_2)} F_1(a_1, b_1, b_2; d; x, y)$$

valid for $R(b_1)>0$, $R(b_2)>0$, $R(c_1-b_1)>0$, $R(c_2-b_2)>0$ and $|x|$, $|y|<1$

$$\int_0^1\int_0^1 u^{c_1-1} v^{c_2-1} (1-u)^{a_1-c_1-1} (1-v)^{a_2-c_2-1} F_2(b, b_1, b_2, c_1, c_2; ux, vy) du\, dv$$

$$= \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(a_1-c_1)\Gamma(a_2-c_2)}{\Gamma(a_1)\Gamma(a_2)} F_2(b, b_1, b_2; a_1, a_2; x, y) \tag{9}$$

valid for $R(c_1)>0$, $R(c_2)>0$, $R(a_1-c_1)>0$, $R(a_2-c_2)>0$ and $|x|+|y|<1$.

$$\int_0^1\int_0^1 u^{a_1-1} v^{a_2-1} (1-u)^{b_1-a_1-1} (1-v)^{b_2-a_2-1} F_3(b_1, b_2; c_1, c_2; d; ux, vy) du\, dv$$

$$= \frac{\Gamma(\mathrm{a}_1)\Gamma(a_2)\Gamma(b_1-a_1)\Gamma(b_2-a_2)}{\Gamma(a_1)\Gamma(a_2)} F_3(a_1, a_2;\ c_1, c_2;\ d;\ x, y) \tag{10}$$

valid for $R(a_1)>0$, $R(a_2)>0$, $R(b_1-a_1)>0$, $R(b_2-a_2)>0$ and $|x|$, $|y|<1$.

$$\int_0^1\int_0^1 u^{a_1-1} v^{a_2-1} (1-u)^{b_1-a_1-1} (1-v)^{b_2-a_2-1} F_4(c_1, c_2; a_1, a_2; ux, vy) du\, dv$$

$$=\frac{\Gamma(a_1)\,\Gamma(a_2)\,\Gamma(b_1-a_1)\,\Gamma(b_2-a_2)}{\Gamma(b_1)\Gamma(b_2)}F_4(c_1,c_2;\ b_1,b_2;\ x,y) \tag{11}$$

valid for $R(a_1)>0$, $R(a_2)>0$, $R(b_1-a_1)>0$, $R(b_2-a_2)>0$, and $|\sqrt{x}|+|\sqrt{y}|<1$.

We give below the proof of (8) and other results can be proved in the similar manner. Let us consider

$$\frac{\partial^{b_1-c_1+b_2-c_2}}{\partial x^{b_1-c_1}\,\partial y^{b_2-c_2}}\,[x^{b_1-1}\,y^{b_2-1}\,F_1(a,c_1,c_2;d;x,y)]$$

$$=\frac{\partial^{b_1-c_1+b_2-c_2}}{\partial x^{b_1-c_1}\,\partial y^{b_2-c_2}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(c_1)_m(c_2)_n}{(d)_{m+n}\,m!\,n!}\,x^{m+b_1-1}\,y^{n+b_2-1}$$

using (5) the right side

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(c_1)_m(c_2)_n}{(d)_{m+n}\,m!\,n!}\,x^{m+c_1-1}\,y^{n+c_2-1}\,\frac{\Gamma(m+b_1)\,\Gamma(n+b_2)}{\Gamma(m+c_1)\,\Gamma(n+c_2)}$$

$$=\frac{\Gamma(b_1)\,\Gamma(b_2)}{\Gamma(c_1)\,\Gamma(c_2)}\,x^{c_1-1}\,y^{c_2-1}\,F_1\,[a,b_1,b_2;d;x,y]$$

Making use of (3) we have

$$\int_0^x\!\!\int_0^y p^{b_1-1}q^{b_2-1}\,(x-p)^{c_1-b_1-1}(y-q)^{c_2-b_2-1}\,F_1(a,c_1,c_2;d;p,q)\;dp\;dq$$

$$=\frac{\Gamma(b_1)\,\Gamma(b_2)}{\Gamma(c_1)\,\Gamma(c_2)}\,x^{c_1-1}\,y^{c_2-1}\,F_1(a,b_1,b_2;d;x,y)$$

Taking $p=ux$ and $q=vy$, we obtain (8)

3. The following results are to be proved in this section

$$\int_0^1\!\!\int_0^1 u^{\rho_2-1}\,v^{\rho_4-1}\,(1-u)^{\beta_1-\beta_2-\rho_2+\rho_1-1}\,(1-v)^{\beta_3-\beta_4-\rho_4+\rho_3-1}$$

$$\times{}_2F_1(\beta_1-\beta_2,\ \beta_2-\rho_1;\ \beta_1-\beta_2-\rho_2+\rho_1;\ 1-u)$$

$$\times{}_2F_1(\beta_3-\beta_4,\ \beta_4-\rho_3;\ \beta_3-\beta_4-\rho_4+\rho_3;\ 1-v)\times F_1(ux,vy)du\ dx$$

$$=\frac{\Gamma(\beta_2)\Gamma(\beta_4)\Gamma(\rho_2)\Gamma(\rho_4)\Gamma(\beta_3-\beta_4-\rho_4+\rho_3)\Gamma(\beta_1-\beta_2-\rho_2+\rho_3)}{\Gamma(\rho_1)\;\Gamma(\rho_3)\;\Gamma(\beta_1)\;\Gamma(\beta_3)}$$

$$F_1(\alpha,\beta_1,\beta_2;\ \gamma;\ x,y) \tag{12}$$

valid for $R(\beta_2)>0$, $R(\beta_4)>0$, $R(\rho_2)>0$, $R(\rho_4)>0$,

$R(\beta_3-\beta_4-\rho_4+\rho_3)>0$, $R(\beta_1-\beta_2-\rho_2+\rho_1)>0$, and $|x|$, $|y|<1$

where

$$F_1(ux,vy)=F\left(\begin{array}{c|c|c} 1 & \alpha & \\ 2 & \beta_1,\rho_1;\beta_2,\rho_3 & ux,vy \\ 1 & \gamma & \\ 1 & \rho_2;\rho_4 & \end{array}\right) \tag{13}$$

$$\int_0^1\int_0^1 u^{\lambda_2-1}\, v^{\lambda_4-1}\,(1-u)^{\rho_1-\gamma_1-\lambda_2+\lambda_1-1}\,(1-v)^{\rho_2-\gamma_2-\lambda_4+\lambda_3-1}$$

$$\times{}_2F_1(\lambda_1-\gamma_1,\ \rho_1-\gamma_1;\ \rho_1-\gamma_1-\lambda_2+\lambda_1;1-u){}_2F_1(\lambda_3-\gamma_2,\ \rho_2-\gamma_2;\ \rho_2-\gamma_2-\lambda_4+\lambda_3;1-v).$$

$$\times F_2(ux,\ vy)\ du\ dv$$

$$=\frac{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\lambda_2)\Gamma(\lambda_4)}{\Gamma(\rho_1)\Gamma(\rho_2)}\ \frac{\Gamma(\rho_1-\gamma_1-\lambda_2+\lambda_1)\Gamma(\rho_2-\gamma_2-\lambda_4+\lambda_3)}{\Gamma(\lambda_1)\Gamma(\lambda_3)}$$

$$\times F_2(\alpha,\beta_1,\beta_2;\ \rho_1,\rho_2;\ x,y) \tag{14}$$

valid for $R(\gamma_1)>0$, $R(\gamma_2)>0$, $R(\lambda_2)>0$, $R(\lambda_4)>0$, $R(\rho_1-\gamma_1-\lambda_2+\lambda_1)>0$, $R(\rho_2-\gamma_2-\lambda_4+\lambda_3)>0$ and $|x|+|y|<1$

where

$$F_2(ux,vy)=F\left(\begin{array}{c|c|c} 1 & \alpha & \\ 2 & \beta_1,\lambda_1;\ \beta_2,\lambda_3 & ux,vy \\ 0 & \cdots\cdots\cdots & \\ 2 & \gamma_1,\lambda_2;\ \gamma_2,\lambda_4 & \end{array}\right) \tag{15}$$

$$\int_0^1\int_0^1 u^{\lambda_2-1}\, v^{\lambda_4-1}\,(1-u)^{\alpha_1-\rho_1+\lambda_1-\lambda_2-1}\,(1-v)^{\alpha_2-\rho_2-\lambda_4+\lambda_3-1}$$

$$\times{}_2F_1(\lambda_1-\rho_1,\alpha_2-\rho_1;\ \alpha_1-\rho_1-\lambda_2+\lambda_1;\ 1-u){}_2F_1(\lambda_3-\rho_2,\alpha_2-\rho_2;\ \alpha_2-\rho_2-\lambda_4+\lambda_3;\ 1-v)$$

$$\times F_3(ux,vy)\ du\ dv=\frac{\Gamma(\rho_1)\Gamma(\rho_2)\Gamma(\lambda_2)\Gamma(\lambda_4)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}$$

$$\frac{\Gamma(\alpha_1-\rho_2-\lambda_4+\lambda_3)\ \Gamma(\alpha_2-\rho_1-\lambda_2+\lambda_1)}{\Gamma(\lambda_1)\ \Gamma(\lambda_3)}\ F_3(\rho_1,\rho_2,\beta_1,\beta_2;\ \gamma;x,y) \tag{16}$$

valid for $R(\rho_1)>0$, $R(\rho_2)>0$, $R(\lambda_2)>0$, $R(\lambda_4)>0$, $R(\alpha_2-\rho_2-\lambda_4+\lambda_3)>0$, $R(\alpha_1-\rho_1-\lambda_2+\lambda_1)>0$, and $|x|$, $|y|<1$

where

$$F_3(ux,\ vy)=F\begin{pmatrix} 0 & & \\ 3 & \alpha_1,\beta_1,\lambda_1,\alpha_2,\beta_2,\lambda_3 & \\ 1 & \gamma_1 & ux,\ vy \\ 1 & \lambda_2;\ \lambda_4 & \end{pmatrix} \tag{17}$$

$$\int_0^1\int_0^1 u^{\lambda_2-1}\, v^{\lambda_4-1}\,(1-u)^{\rho_1-\gamma_1-\lambda_2+\lambda_1-1}\,(1-v)^{\rho_2-\gamma_2-\lambda_4+\lambda_3-1}$$

$$\times{}_2F_1(\lambda_1-\gamma_1,\rho_1-\gamma_1;\rho_1-\gamma_1-\lambda_2+\lambda_1;\ 1-u)\ {}_2F_1(\lambda_3-\gamma_2;\ \rho_2-\gamma_2;\rho_2-\gamma_2-\lambda_4+\lambda_3;\ 1-v)$$

$$F_4(ux,vy)\ du\ dv=\frac{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\lambda_2)\Gamma(\lambda_4)\Gamma(\rho_1-\gamma_1-\lambda_2+\lambda_1)\Gamma(\rho_2-\gamma_2-\lambda_4+\lambda_3)}{\Gamma(\rho_1)\Gamma(\rho_2)\Gamma(\lambda_1)\Gamma(\lambda_3)}$$

$$\times F_4(\alpha,\beta;\ \rho_1,\rho_2;\ x,y) \tag{18}$$

valid for $R(\gamma_1)>0$, $R(\gamma_2)>0$, $R(\lambda_2)>0$, $R(\lambda_4)>0$ $R(\rho_1-\gamma_1-\lambda_2+\lambda_1)>0$, $R(\rho_2-\gamma_2-\lambda_4+\lambda_3)>0$, $|\sqrt{x}|+|\sqrt{y}|<1$ where

$$F_4(ux,vy)=F\begin{pmatrix} 2 & \alpha,\ \beta & \\ 1 & \lambda_1;\ \lambda_3 & \\ 0 & \cdots\cdots\cdots & ux,\ vy \\ 2 & \gamma_1,\ \lambda_2;\ \gamma_2,\ \gamma_4 & \end{pmatrix} \tag{19}$$

We shall give the proof of (12) only and other results can be proved in the similar way.

Let us consider

$$\frac{\partial^{\rho_2-\rho_1+\rho_4-\rho_3}}{\partial u^{\rho_2-\rho_1}\ \partial v^{\rho_4-\rho_3}}\ [u^{\rho_2-1}\ v^{\rho_4-1}\ F_1(ux,vy)]$$

$$=\frac{\partial^{\rho_2-\rho_1+\rho_4-\rho_3}}{\partial u^{\rho_2-\rho_1}\ \partial v^{\rho_4-\rho_3}}\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta_1)_m(\beta_2)_n(\rho_1)_m(\rho_3)_n}{(\gamma)_{m+n}(\rho_2)_m(\rho_4)_n\ m!\ n!}\ x^m\ y^n\right]$$

$$=\frac{u^{\rho_1-1}\ v^{\rho_3-1}\ \Gamma(\rho_2)\Gamma(\rho_4)}{\Gamma(\rho_1)\ \Gamma(\rho_3)}\ F_1(\alpha,\beta_1,\beta_2;\ \gamma,ux,vy) \tag{20}$$

Substituting the value of F_1 from (20) in (8), we get

$$\int_0^1\int_0^1 u^{\beta_2-\rho_1}\ v^{\beta_4-\rho_3}\ (1-u)^{\beta_1-\beta_2-1}\ (1-v)^{\beta_3-\beta_4-1}$$

$$\times\frac{\partial^{\rho_2-\rho_1+\rho_4-\rho_3}}{\partial u^{\rho_2-\rho_1}\ \partial u^{\rho_4-\rho_3}}\ [u^{\rho_2-1}\ v^{\rho_4-1}\ F_1(ux,vy)]\ du\ dv$$

$$=\frac{\Gamma(\beta_2)\Gamma(\beta_4)\Gamma(\rho_2)\Gamma(\rho_4)\Gamma(\beta_1-\beta_2)\Gamma(\beta_3-\beta_4)}{\Gamma(\rho_1)\Gamma(\rho_3)\Gamma(\beta_1)\Gamma(\beta_3)} F_1(\alpha,\beta_2,\beta_4;\gamma;x,y).$$

Now using the relation (1), we obtain

$$\int_0^1\int_0^1 u^{\rho_2-1} v^{\rho_4-1} F_1(ux,vy) \frac{\partial^{\rho_2-\rho_1}}{\partial(1-u)^{\rho_2-\rho}} [u^{\beta_2-\rho_1}(1-u)^{\beta_1-\beta_2-1}]$$

$$\frac{\partial^{\rho_4-\rho_3}}{\partial(1-v)^{\rho_4-\rho_3}} [v^{\beta_4-\rho_3}(1-v)^{\beta_3-\beta_4-1}]\, du\, dv$$

$$=\frac{\Gamma(\beta_2)\Gamma(\beta_4)\Gamma(\rho_2)\Gamma(\rho_4)\Gamma(\beta_1-\beta_2)\Gamma(\beta_3-\beta_4)}{\Gamma(\rho_1)\Gamma(\rho_3)\Gamma(\beta_1)\Gamma(\beta_3)} F_1(\alpha,\beta_2,\beta_4;\gamma;x,y).$$

Using (6) we obtain (12).

4. The following double integrals given in this section can be obtained from the definitions of Appell's functions [**1**].

$$\int_0^\infty\int_0^\infty x^{\lambda-1} y^{\mu-1} e^{-(x+y)} {}_1F_1(\alpha,\beta;ux+vy)\, dx\, dy$$

$$=\Gamma(\lambda)\Gamma(\mu)F_1(\alpha,\lambda,\mu;\beta:u,v) \tag{21}$$

valid for $R(\lambda)>0$, $R(\mu)>0$, $|u|$, $|v|<1$

$$\int_0^\infty\int_0^\infty x^{\lambda-1} y^{\mu-1} e^{-(x+y)} {}_1F_1(\rho;\gamma;\ \alpha x)\ {}_0F_1(\delta;\beta xy)\, dx\, dy$$

$$=\Gamma(\lambda)\Gamma(\mu)F_2(\lambda,\rho,\mu;\gamma,\delta;\alpha,\beta) \tag{22}$$

valid for $R(\lambda)>0$, $R(\mu)>0$, $|\alpha|+|\beta|<1$

$$\int_0^\infty\int_0^\infty x^{2\lambda-1} y^{2\mu-1} k_{2v}(x)k_{2\rho}(y)\ {}_0F_1(\gamma;\ \alpha x^2+\beta y^2)\, dx\, dy$$

$$=\frac{\Gamma(\lambda\pm v)\Gamma(\mu\pm\rho)}{2^{4-2\lambda-2\mu}} F_3(\lambda+v,\ \mu+\rho,\ \lambda-v,\ \mu-\rho;\ \gamma;\ \alpha,\beta). \tag{23}$$

valid for $R(\lambda\pm v)>0$, $R(\mu\pm\rho)>0$, $|\alpha|$, $|\beta|<1$

$$\int_0^\infty\int_0^\infty x^{\lambda-1} y^{\mu-1} e^{-(x+y)} {}_0F_1(\rho_1;\ a^2xy)\ {}_0F_1(\rho_2;\ b^2xy)\, dx\, dy$$

$$=\Gamma(\lambda)\Gamma(\mu)F_4(\lambda,\mu;\ \rho_1,\rho_2;\ a^2,b^2) \tag{24}$$

validfor $R(\lambda)>0$, $R(\mu)>0$, $|a|+|b|<1$.

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