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ZEROS AND ORTHOGONALITY OF THE POLYNOMIAL SET $B_n(x,y)$

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1. Introduction

The present paper comprises of a discussion on the nature of the zeros of the polynomial set $B_n(x,y)$ followed by a note on its orthogonality. $B_n(x,y)$, the generalization of as many as eighteen classical polynomials like Hermite polynomial, Laguerre polynomial, Legendre polynomial etc. have been defined by means of the generating relation

$$\sum_{n=0}^{\infty} B_n(x,y)\ t^n = {}_pF_q[(\alpha_p);(\beta_q);\nu xt]\ {}_rF_s[(a_r);(b_s);\mu y^{-m}t^m] \qquad (1.1)$$

valid under suitable conditions given in [1] and [2].

2. Zeros of $B_n(x,y)$

By the use of 'Descartes' Rule of signs of the theory of equation [4], the following conclusions regarding the zeros of the polynomial set $B_n(x,y)$ can be summarized as:

A. If ν, μ, y, (α_p), (β_q), (a_r) and (b_s) be all positive, then

(i) $B_n(x,y)$ has no positive zero.

(ii) It has no negative zero if m be even

(iii) If m be o d, the number of negative zeros is not more than $\left[\frac{n}{m}\right]$.

(iv) If n be not an integral multiple of m, then $x=0$ is at least one of the zeros, while otherwise, results, (i) to (iii) again hold good.

B. If ν, y, (α_p), (β_q), (a_r) and (b_s) be all positive, but μ be negative then

(i) Numbers of positive zeros of $B_n(x,y)$ are not more than $\left[\frac{n}{m}\right]$.

(ii) It has no negative zero, if m be odd.

(iii) If m be even, maximum number of its negative zeros is $\left[\frac{n}{m}\right]$

(iv) Same as (iv) of (A).

C, If μ, y, (α_p), (β_q), (a_r) and (b_r) be all positive, but ν be negative, then according as m is even or odd, all the four conclusions of (A) and (B) respectively hold good.

D. If μ, ν, (α_p), (β_q), (a_r) and (b_s) be all positive, but y be negative then according as m is even or odd, all the four conclusions of (A) and (B) respectively hold good.

E. If y, (α_p), (β_q), (a_r) and (b_s) be all positive, but μ and ν be both negative, then according as m is odd or even, (A) and (B) respectively hold good.

F. If ν, (α_p), (β_q), (a_r) and (b_s) be all positive, but μ and y be both negative, then according as m is odd or even, (A) and (B) respectively hold good.

G. If μ, (α_p), (β_q), (a_r) and (b_s) be all positive, but ν and y be both negative, then all the conclusions of (A) hold good.

H. If (α_p), (β_q), (a_r) and (b_s) be all positive, but ν, μ and y be all negative, then all the conclusions of (B) hold good.

I. Furthermore, by virtue of several examples similar to the two given below, it is evident that the zeros of $B_n(x,y)$ are not all distinct:

EXAMPLE 1. $B^{1;\mu:(\alpha_p);(a_r)}_{2;\nu;(\alpha_p);(a_r)}(x,y)=\frac{1}{2y^2}(\nu xy+\mu)^2.$

EXAMPLE 2. $B^{1;\mu;-1/2,-1/2;(a_r)}_{2;\nu;\text{———};(a_r)}(x,y)=\frac{1}{8y^2}(\nu xy+4\mu)^2.$

J. The zeros of $B_n(x,y)$ are not interlaced.

3. Orthogonality

The polynomial set $B_n(x,y)$ is not orthogonal, which is evident because of the following observations:

I. we know that, if a simple set of real polynomials $\phi_n(x)$ is orthogonal, then the zeros of $\phi_n(x)$ are all distinct [5; p. 149. theorem, 55]. But in our case, res ults of article 2 show that the zeros of $B_n(x,y)$ is not orthognal.

II. Recently, J. Shohat [6] has shown that a necessary and sufficient condition, that the sequence of polynomials

$$\phi_0(x)=1$$
$$\phi_1(x)=x-c_1$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\phi_n(x)=x^n-S_nx^{n-1}+d_{n,n-2}x^{n-2}+\cdots\cdots(n=2,3,\ \cdots\cdots)$$

form a sequence of orthogonal polynomial, is that they satisfy a differential equation of the form

$$\phi_n(x)=(x-c_n)\phi_{n-1}(x)-\lambda_n\phi_{n-2}(x), \quad (n\geq 2; c_n, \lambda_n \text{ are constants})$$

with positive λ_n.

Now, our polynomial set $B_n(x,y)$ does not satisfy this type of difference equation. Hence it is not orthogonal.

III. Abdul Halim and W. A. Al-Salam [3] have given two elegent proofs of the result, that the only orthogonal polynomials of the form

$$(3.1) \qquad {}_{p+1}F_q\left[\begin{matrix}-n, & (a_p) \\ & (b_q)\end{matrix}; x\right],$$

where n is a non-negative integer and the a_i's and b_j's are independent of x and n. are ${}_1F_1[-n; b; x]$.

Hence, clearly, our polynomial, which is a generalization of (3.2) cannot be orthogonal.

IV. Since our polynomial set does not satisfy the three-term recurrence relation, we observe [5; p. 151(1)] that $B_n(x,y)$ is not orthogonal.

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REFERENCES

[1] Agrawal, B. D. & Khanna, I. K. *A general differential equations for classical, polynomials*, Proc. Amer. Math. Soc. vol. 22, 1969, 646–649.

[2] Agrawal, B. D. & Khanna, I. K., *Unification of certain classical polynomial*, Prog. of Math. vol. 8, no. 2, 1974, p. 15–24.

[3] Abdual-Halim, N and Al-Salim, *A characterization of the Laguerre polynomials*, Rend. Sem. Univ. Padova, 34 (1964), 176–179.

[4] Hall and Knight, *Higher Algebra.*

[5] Rainville, E. D., *Special functions*, Macmillan, New York, 1960.

[6] Shohat, J., *The relation of the classical orthogonal polynomials to the polynomials of Appell*, Amer. Jour. of Math., vol.58 (1936), 453–464.