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A GENERALIZATION OF KARLSSONS' FORMULA

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1. Introduction

In a recent paper [3], Karlsson has proved the formula

$$ {}_pF_q\left[\begin{matrix} b_1+m_1, \cdots, b_n+m_n, a_{n+1}, \cdots, a_p; Z \\ b_1, \cdots, b_n, b_{n+1}, \cdots, b_q; \end{matrix}\right] $$

$$ = \sum_{J_1=0}^{m_1} \cdots \sum_{J_n=0}^{m_n} A(J_1, \cdots, J_n) Z^{J_n} {}_{p-n}F_{q-n} \left[\begin{matrix} a_{n+1}+J_n, \cdots, a_p+J_n; Z \\ b_{n+1}+J_n, \cdots, b_q+J_n; \end{matrix}\right], \tag{1} $$

where

$$ J_n = J_1 + \cdots + J_n, \tag{2} $$

$$ A(J_1, \cdots, J_n) = \binom{m_1}{J_1} \cdots \binom{m_n}{J_n} \frac{(b_2+m_2)_{J_1}(b_3+m_3)_{J_2} \cdots (b_n+m_n)_{J_{n-1}}}{(b_1)_{j_1}(b_2)_{j_2} \cdots (b_n)_{j_n}} $$

$$ \times \frac{(a_{n+1})_{J_n} \cdots (a_p)_{J_n}}{(b_{n+1})_{J_n} \cdots (b_q)_{J_n}} \tag{3} $$

and

$$ (c)_r = \frac{\Gamma(c+r)}{\Gamma(c)}. \tag{4} $$

According to Erdelyi [1] the fractional derivative is defined by

$$ D_x^\lambda [x^{\mu-1}] = \frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)} x^{\mu-\lambda-1}, \tag{5} $$

which holds for all values of λ except $\lambda=\mu$. By means of (5), we can easily establish the formula

$$ D_x^\mu [x^{\mu+\rho-1} {}_pF_q(\alpha_p; \beta_q; x)] $$

$$ = \frac{\Gamma(\mu+\rho)}{\Gamma(\rho)} x^{\rho-1} {}_{p+1}F_{q+1}(\alpha_p, \mu+\rho; \beta_q, \rho; x). \tag{6} $$

In the investigation we also require the following theorem due to author and Manocha [2].

THEOREM. *If U and V are analytic functions of x, then*

$$ D_x^\lambda [UV] = \sum_{n=0}^{\infty} \binom{\lambda}{n} D_x^{\lambda-n}(U) D_x^n (V), \tag{7} $$

where λ is a complex number.

2. The formula to be proved is

$$
{}_{p+m}F_{q+m}\begin{bmatrix}\mu_1+\rho_1, & \cdots, & \mu_m+\rho_m, & \alpha_p; & x\\ \mu_1, & \cdots, & \mu_m, & \beta_q; & \end{bmatrix}
$$

$$
=\sum_{r_1=0}^{\infty}\cdots\sum_{r_m=0}^{\infty}A(r_1,\ \cdots,\ r_m)x^{R_m}\,{}_pF_q\begin{bmatrix}\alpha_p+R_m; & x\\ \beta_q+R_m; & \end{bmatrix}, \tag{8}
$$

where

$$
R_m=r_1+\cdots+r_m, \tag{9}
$$

$$
A(r_1,\ \cdots,\ r_m)=\frac{(\alpha_p)R_m(\mu_2+\rho_2)R_1\cdots(\mu_m+\rho_m)R_{m-1}}{(\beta_q)_{R_m}(\mu_1)_{R_1}(\mu_2)_{R_2}\cdots(\mu_m)_{R_m}}\binom{\rho_1}{r_1}\cdots\binom{\rho_m}{r_m} \tag{10}
$$

By the principle of Analytic continuation, Eq. (8) is valid whenever the functions involved are all analytic.

PROOF. To prove (8), we take $V=x^{\mu+\rho-1}$ and $V={}_pF_q(\alpha_p;\ \beta_q;\ x)$, substituting these values in (7) and using (5) and (6), we have

$$
{}_{p+1}F_{q+1}[\alpha_p,\ \mu+\rho;\ \beta_q,\ \rho;\ x]
$$

$$
=\sum_{n=0}^{\infty}\binom{\mu}{n}\frac{(\alpha_p)_n\ x^n}{(\beta_q)_n\ (\rho)_n}\ {}_pF_q\ [\alpha_p+n;\ \beta_q+n;\ x] \tag{11}
$$

we repeat this process to m times, we get (8). This completes the proof.
In particular, if we take $\rho_1=m_1,\ \cdots,\ \rho_m=m_m$, where $m_1,\ \cdots,\ m_m$ are positive integers, in (8), it reduces to (1).

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