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A THEOREM ON C-LOOPS

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1. Introduction

F. Fenyves [2], [3] defined extra loops and C-loops. He showed the equivalence of extra loops and Moufang loops.

The purpose of the present note is to obtain the equivalence between extra loops and C-loops.

2. Preliminary information

Let $(G,.)$ be a loop whose identity element is denoted by 1. For each $x \in G$ the mappings $R(x)$ and $L(x)$ are defined by $y\,R(x)=yx$ and $y\,L(x)=xy$ for all $y \in G$. It follows that $R(x)$ and $L(x)$ are one-to-one mappings of G onto G. If $(G,.)$ is an inverse property loop, then corresponding to each $x \in G$ there is an $xJ=x^{-1} \in G$ so that $xx^{-1}=x^{-1}x=1$, $R(x)^{-1}=R(x^{-1})$ and $L(x)^{-1}=L(x^{-1})$. Further, $J^2=I$, $JL(x)J=R(x)^{-1}$, $JR(x)J=L(x)^{-1}$. J is known as inverse mapping. The left nucleus N_λ, the middle nucleus N_μ, and the right nucleus N_ρ of a loop $(G,.)$ are defined by

$$N_\lambda=\{\text{all } x \in G \mid x\cdot yz=xy\cdot z, \text{ all } y, z \in G\},$$
$$N_\mu=\{\text{all } y \in G \mid x\cdot yz=xy\cdot z, \text{ all } x, z \in G\}$$
$$N_\rho=\{\text{all } z \in G \mid x\cdot yz=xy\cdot z, \text{ all } x, y \in G\}$$

The nucleus N of $(G,.)$ is $N=N_\lambda \cap N_\mu \cap N_\rho$.

Recall that an ordered triple (U, V, W) of one-to-one mappings U, V and W of G onto G is called an autotopism of the loop $(G,.)$ if and only if $xU\cdot yV=(xy)W$ for all $x, y \in G$. It is well known that the set of all autotopisms of a loop forms a group under the usual "componentwise multiplication".

For a detailed account of the loop theory concepts mentioned in the preceding paragraphs see Bruck [1]. We summarize those basic results needed about Moufang loops and in particular, about extra loops and inverse property loops as follows:

THEOREM 1. *Let $(G,.)$ be a Moufang loop. Then*

(i) $(G,.)$ *is an inverse property loop* (see Bruck [1], Ch. VII, Lemma 3.1),

(ii) *the right, middle and left nuclei of* $(G,.)$ *coincide with the nucleus of* $(G,.)$ (see Bruck [1], Ch, VII, Theorem 2.1).

DEFINITION 1. A loop $(G,.)$ is *an extra loop* if and only if

(1) $$(xy\cdot z)x=x(y.zx)$$

for all $x,\ y,\ z\in G$. (See Fenyves [2])

THEOREM 2. *Let* $(G,.)$ *be an extra loop. Then*$(G,.)$ *is Moufang.* (See Fenyves [2], Theorem 3).

THEOREM 3. *A loop* $(G,.)$ *is an extra loop if and only if*

$$A_1(x)=(L(x),\ R(x)^{-1},\ L(x)R(x)^{-1})$$

is an autotopism of $(G,.)$ *for all* $x\in G$ (See Fenyves [2], Theorem 2).

THEOREM 4. *If* (U,V,W) *is an autotopism of an inverse property loop* $(G,.)$. *Then* (JUJ,W,V) *and* (W,JVJ,U) *are also the autotopisms of* $(G,.)$, (see Bruck [1], Ch VII Lemma 2.1), *where* $J:x\longrightarrow x^{-1}$ *is the inverse mapping of* G.

DEFINITION 2. A loop $(G,.)$ is *a C-loop* if and only if

(2) $$(yx\cdot x)z=y(x\cdot xz)$$

for all $x,y,\ z\in G$. (See Fenyves [3]).

THEOREM 5. *Let* $(G,.)$ *be a C-loop. Then* $(G,.)$ *has inverse property.* (See Fenyves [3], Proof of Theorem 4).

3. Main Theorem

First we prove the following lemma:

LEMMA. *A loop* $(G,.)$ *is a C-loop if and only if* $(R(x)R(x),\ L(x)^{-1}L(x)^{-1},\ I)$ *is an autotopism of* G, *where* I *is the identity mapping of* G.

PROOF. $(G,.)$ is a C-loop if and only if (2) holds i.e. $(yx\cdot x)z=y(x\cdot xz)$ for all $x,\ y,\ z\in G$. Replacing z by $x^{-1}(x^{-1}z)$ and appealing to the fact that a C-loop is an inverse property loop (See Theorem 5), we have, equivalently, that $(G,.)$ is C-loop if and only if

$$(yx\cdot x)(x^{-1}(x^{-1}z))=yz$$
$$\Longleftrightarrow yR(x)R(x)\cdot zL(x)^{-1}L(x)^{-1}=(yz)I \quad \text{for all } y,z\in G.$$
$$\Longleftrightarrow (R(x)R(x),L(x)^{-1}L(x)^{-1},\ I)$$

is an autotopism of G and the proof is complete.

We are now in a position to present the main

THEOREM. *The following statements are equivalent for a loop* $(G,.)$:

(a) $(G,.)$ *is an extra loop,*

(b) $(G,.)$ *is C-loop and* $(R(x)^{-2}L(x),\ L(x)^2R(x)^{-1},\ L(x)R(x)^{-1})$ *is an autotopism of* $(G,.)$ *for all* $x\in G$.

PROOF. (*a*) $\Longrightarrow$ (*b*). Suppose $(G,.)$ is an extra loop. Then from Theorem 3 we have the the autotopism

$$A_1(x)=(L(x),\ R(x)^{-1},\ L(x)R(x)^{-1}).$$

Also from Theorem 2, $(G,.)$ is Moufang i.e. satisfies the following Moufang identity:

(3) $$(xy)\ (zx)=x[(yz)\cdot x]$$

Now (3) gives the autotopism $A_2(x)=(L(x), R(x), R(x)L(x))$ of $(G,.)$. Therefore, we have the autotopism

$$\begin{aligned}A_1(x)A_2(x)&=(L(x), R(x)^{-1},\ L(x)R(x)^{-1})(L(x),\ R(x),\ R(x)L(x))\\&=(L(x)L(x),\ I,\ L(x)\ L(x)).\end{aligned}$$

Then by Theorem 4, $A_3(x)=(JL(x)L(x)J, L(x)L(x),\ I)=(R(x)^{-1}R(x)^{-1}, L(x)L(x),$ $I)$ is an autotopism of $(G,.)$. This implies that $(R(x)^{-1}R(x)^{-1},\ L(x)L(x),$ $I)^{-1}=(R(x)\ R(x),\ L(x)^{-1}L(x)^{-1}, I)$ is an autotopism of $(G,.)$.

Hence, by lemma $(G,.)$ is a C-loop. Also, $A_3(x)A_1(x)=(R(x)^{-2}, L(x)^2, I)(L(x),$ $R(x)^{-1},\ L(x)R(x)^{-1})=(R(x)^{-2}L(x),\ L(x)^2R(x)^{-1},\ L(x)R(x)^{-1})$ is an autotopism of $(G,.)$. Hence, (a) $\Longrightarrow$ (b).

(b) $\Longrightarrow$ (a). Assume (b). Then $(R(x)^{-2}L(x),\ L(x)^2R(x)^{-1},\ L(x)R(x)^{-1})$ and by lemma, $(R(x)^2,\ L(x)^{-2},\ I)$ are both the autotopisms of $(G,.)$. Combining both the autotopisms we have that $(L(x),\ R(x)^{-1},\ L(x)R(x)^{-1})$ is an autotopism of $(G,.)$. Therefore, byTheorem 3, $(G,.)$ is extra loop and the proof is complete.

REMARK. As alternative law holds in a C-loop, we have

$$R(x)^2=R(x^2),\ R(x)^{-2}=R(x^2)^{-1},\ L(x)^2=L(x^2).$$

Therefore, if the mapping $x\longrightarrow x^2$ is a permutation of a loop $(G,.)$, then the above theorem can be stated as follows:

The following statements are equivalent for a loop $(G,.)$:

(a′) $(G,.)$ is an extra loop,

(b′) $(G,.)$ is C-loop and $(R(x)^{-1}L(x), L(x)R(x)^{-1}, L(x)R(x)^{-1})$ is an autotopism of $(G,.)$ $\forall x \in G$.

COROLLARY. *If the mapping $x \longrightarrow x^2$ is a permutation of an extra loop $(G,.)$, then $(G,.)$ is a group.*

PROOF. If $(G,.)$ is an extra loop, then by main Theorem, $(G,.)$ is C-loop. Therefore, (2) holds and by alternative laws, we have

$$(yx^2)z=y(x^2z) \ \forall y, z \in G.$$

This implies that x^2 is in the middle nucleus (N_μ) of $(G,.)$. Using Therem 1 and Theorem 5, we have x^2 in the nucleus of $(G,.)$. As $x \longrightarrow x^2$ is permutation, we can say that x is in the nucleus i.e. x associates with every element of $(G,.)$ $\forall x \in G$. Hence, $(G,.)$ is associative. This implies $(G,.)$ is a group.

This completes the Corollary.

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