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SOME APPROACHES TO CLUSTER SETS IN GENERAL TOPOLOGY

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The study of cluster sets was first introduced by Painlevé in his lectures in Stockholm in 1895 [2, page 1], and has been extensively developed since that time for complex valued functions of a complex variable. Several definitions have recently appeared in the literature of sets which are extensions of the notion of cluster set to arbitrary topological spaces. The purpose of this paper is to unify the theory of cluster sets in general topology and to prove some theorems giving sufficient conditions for a function with a closed graph to be continuous.

Hereafter, we will use $\mathrm{Cl}(A)$ to denote the closure of A, A' to denote the set of limit points of A, $N(X)$ to denote the neighborhood system at a point x, and X and Y will be used to denote arbitrary topological spaces with additional hypotheses stated as needed.

1. Cluster sets

Consider the following definitions.

DEFINITION 1 [8]. Let $f:X\to Y$, $x\in X$, with X and Y Hausdorff. The *set of limit points of f at x*, denoted $L(f\,;x)$, is the set of all points y in Y such that there exists a sequence $x_n\to x$ with $f(x_n)\to y$.

DEFINITION 2 [6]. Let $f:X\to R$ where R denotes the real line. The *H-cluster set of f at* $x\in X$, denoted $HC(f\,;x)$, is the set of all real numbers y in R such that for each $r>0$, $x\in [f^{-1}((y-r,\ y+r))]'$.

DEFINITION 3 [11]. Let D denote the domain of a function $f:X\to Y$. If $x\in X$, the *W-cluster set of f at x*, denoted $WC(f\,;X)$, is defined to be

$$WC(f\,;x)=\bigcap\{\mathrm{cl}(f(U\cap D)) : U\in N(x)\}.$$

DEFINITION 4 [1]. Let $f:X\to Y$, $x\in X$. The set $D(f\,;x)$ consists of all points $y\in Y$ such that $f(U)\cap V\neq\phi$ for all $U\in N(x)$ and $V\in N(y)$.

In order to simplify our discussion, we will only consider functions $f:X\to Y$ where f is defined on all of X, and we will ignore the Hausdorff restrictions placed on X and Y in Definition 1. It will be convenient to have the following extension of Definition 2.

DEFINITION 2′ [5]. Let $f: X \to Y$, $x \in X$. The *H-cluster set of f at x*, denoted $HC(f; x)$, is the set of all $y \in Y$ such that $x \in [f^{-1}(V)]'$ for every $V \in N(y)$.

The next two theorems give some useful characterizations of Definitions 3 and 2′.

THEOREM 1.1. *Let $f: X \to Y$, $x \in X$. Then the following are equivalent.*

(1) $y \in WC(f; x)$.

(2) $f^{-1}(N(y))$ *accumulates at* x.

(3) $f(N(x))$ *accumulates at* y.

(4) *There exists a filter* $\mathscr{F}$ *on* X *such that* $\mathscr{F} \to x$ *and* $f(\mathscr{F}) \to y$.

(5) *There exists a net* $x_\alpha \to x$ *such that* $f(x_\alpha) \to y$.

PROOF. The equivalence of (1), (2), and (4) is shown in Theorem 2.2 of [4]. The equivalence of (2) and (3) is trivial, and the equivalence of (4) and (5) is well known [3, page 213].

THEOREM 1.2 [5, Theorem 4.2]. *Let $f: X \to Y$, $x \in X$. Then the following are equivalent.*

(1) $y \in HC(f; x)$.

(2) *There exists a filterbase* $\mathscr{B}$ *on* $X \setminus \{x\}$ *such that* $\mathscr{B} \to x$ *and* $f(\mathscr{B}) \to y$.

(3) $y \in \cap \{\mathrm{Cl}(f(U \setminus \{x\})) : U \in N(x)\}$.

The proof of the following theorem is straightforward and hence omitted.

THEOREM 1.3. *Let $f: X \to Y$, $x \in X$.*

(1) $WC(f; x) = D(f; x)$.

(2) $L(f; x) \subset WC(f; x)$.

(3) $HC(f; x) \subset WC(f; x)$.

(4) *If X and Y are first countable, then* $L(f; x) = WC(f; x)$.

Note that (4) and (1) of the above theorem show Theorems 6 and 7 of [1] are merely restatements of Theorems 3.7 and 3.8 respectively of [8].

THEOREM 1.4. *If $f: X \to Y$, $x \in X$, and Y is T_1, then* $WC(f; x) \setminus HC(f; x) \subset \{f(x)\}$.

PROOF. Assume $y \in WC(f; x) \setminus HC(f; x)$. If we suppose $y \neq f(x)$, then there must exist $V \in N(y)$ and $U \in N(x)$ such that $f^{-1}(V) \cap (U \setminus \{x\}) = \phi$, but $f^{-1}(V) \cap U \neq \phi$. This implies $f^{-1}(V) \cap U = \{x\}$. Since Y is T_1 and $y \neq f(x)$, there exists a $W \in N(y)$ such that $f(x) \notin W$. Hence $x \notin f^{-1}(W)$ and $\{x\} \not\subset f^{-1}(W \cap V)$. This implies $f^{-1}(W\cap V)\cap U=\phi$ and $y\notin WC(f\,;\,x)$ which is a contradiction.

A function $f: X\to Y$ is said to be ***connected*** [8] if $f(C)$ is connected in Y for every connected set C in X.

THEOREM 1.5. Let $f: X\to Y$ be connected with X locally connected and Y T_1 If $U\backslash\{x\}$ is connected for every $U\in N(x)$, and x is a limit point of X, then $HC(f\,;\,x)=WC(f\,;\,x)$.

PROOF. Theorem 4.4 of [5] shows $f(x)\in HC(f\,;\,x)$, hence $WC(f\,;\,x)\subset HC(f;x)$ by Theorem 1.4. Equality follows from Theorem 1.3, (3).

The following theorem will be useful in what follows.

THEOREM 1.6. [4, Theorem 2.3, (a)] *Let $f: X\to Y$ be connected with X locally connected and Y compact Hausdorff. Then $WC(f\,;\,x)$ is connected for every x in X.*

THEOREM 1.7. *With the same hypotheses as in Theorem* 1.6,

(1) *Either $HC(f\,;\,x)=\phi$ or $HC(f\,;\,x)=WC(f\,;\,x)$.*

(2) *If x is a limit point of X, then $HC(f\,;\,x)=WC(f\,;\,x)$.*

PROOF. To show (1), assume $HC(f;x)\neq\phi$ and observe we need only show $f(x)\in HC(f\,;\,x)$. If we suppose $f(x)\notin HC(f\,;\,x)$, then $WC(f\,;\,x)=HC(f\,;\,x)\cup\{f(x)\}$ is a disconnection of $WC(f\,;\,x)$, which is a contradiction, and we have shown (1).

(2) follows from (1).

2. Functions with closed graphs

In section 2 we will use $C(f\,;\,x)$ to denote $WC(f\,;\,x)$. If $f: X\to Y$, then the *graph of f*, denoted $\mathrm{Gr}(f)$, is the subset $\{(x, f(x)) : x\in X\}$ of $X\times Y$.

Our purpose in this section is to illustrate how cluster set techniques can be used in general topology. Our principal tool will be the following theorem.

THEOREM 2.1. *Let $f: X\to Y$. Then $\mathrm{Gr}(f)$ is closed if and only if $C(f\,;\,x)=\{f(x)\}$ for every $x\in X$.*

PROOF. Necessity. Assume $C(f\,;\,x)=\{f(x)\}$ for every $x\in X$, and let $(x,y)\in \mathrm{Cl}(\mathrm{Gr}(f))$. Then there exists a net $(x_\alpha, f(x_\alpha))$ on $\mathrm{Gr}(f)$ such that $(x_\alpha, f(x_\alpha))\to(x,y)$. This implies $x_\alpha\to x$ and $f(x_\alpha)\to y$. Hence $y\in C(f\,;\,x)$ by Theorem 1.1, (5). We conclude $y=f(x)$. $(x,y)\in \mathrm{Gr}(f)$, and $\mathrm{Gr}(f)$ is closed.

Sufficiency. Assume $\mathrm{Gr}(f)$ is closed and suppose $y\in C(f\,;\,x)$. Then there exists a net $x_\alpha \to x$ such that $f(x_\alpha)\to y$ and hence $(x_\alpha, f(x_\alpha))\to(x,y)$. Since Gr$(f)$ is closed we must have $y=f(x)$.

We will prove the follwing well known theorem [10, Exercise 16.10, page 130] to illustrate cluster set technique.

THEOREM 2.2. *Let $f:X\to Y$ with Y compact. If* Gr(f) *is closed, then f is continuous.*

PROOF. Let $x\in X$ and let V be an open neighborhood of $f(x)$ in Y. Since f has a closed graph, there must exist a $U\in N(x)$ such that $f(U)\cap(Y\backslash V)=\phi$. Indeed, otherwise $\{\mathrm{cl}(f(U))\cap(Y\backslash V) : U\in N(x)\}$ is a collection of closed sets in the compact subspace $Y\backslash V$ which satisfies the finite intersection property. This implies $C(f\,;x)\cap(Y\backslash Y)\neq\phi$, which is a contradiction, and the proof is complete.

We will only remark that if in the above theorem X is first countable and Y is countably compact, a similar technique can be used to show the same conclusion. This is, however, another well known result [7, Theorem 2]. The following result is perhaps not so well known.

THEOREM 2.3. *Let $f : X\to Y$ be an open mapping with X first countable and Y a completely regular pseudocompact space. If* Gr(f) *is closed, then f is continuous.*

PROOF. Suppose f is not continuous at $x\in X$. Let $\{U_n\}$ be a monotone decreasing neighborhood base of open sets at x. There exists some open set V containing $f(x)$ such that $f(U_n)\cap(Y\backslash \mathrm{Cl}(V))\neq\phi$ for every n. Let W be an open neighborhood of $(f(x)$ such that $W\subset\mathrm{Cl}(W)\subset V$ and hence

$$Y\backslash W\supset \mathrm{Cl}(Y\backslash\mathrm{Cl}(W))\supset Y\backslash V\supset Y\backslash\mathrm{Cl}(V).$$

Let $S=\mathrm{Cl}(Y\backslash\mathrm{Cl}(W))$. Observe that S is a completely regelar pseudocompact subspase.

Now $\{f(U_n)\cap(Y\backslash\mathrm{Cl}(V))\}$ is a descending sequence of open sets in S, hence there exists a point $y\in\bigcap_{n=1}^{\infty}\mathrm{Cl}_S[f(U_n)\cap(Y\backslash\mathrm{Cl}(V))]$. It follows from the relation

$$\mathrm{Cl}_S[f(U_n)\cap(Y\backslash\mathrm{Cl}(V))]\subset\mathrm{Cl}(f(U_n))\cap(Y\backslash W)$$

that $y\in C(f\,;x)\cap(Y\backslash W)$, and this contradiction completes the proof.

We conclude with the following application to functional analysis. The terminology is the same as that used in [9].

THEOREM 2.4. Let X and Y be topological vector spaces whose topologies are induced by a complete invariant metric, and let $\Lambda : X\to Y$ be a linear map which maps closed subsets of X onto closed subsets of Y. Then Λ is continuous if and only if $N(\Lambda)=\Lambda^{-1}\{0\}$ is closed.

PROOF. Necessity is obvious. To show sufficiency, observe that $N(\Lambda)$ closed implies $\Lambda^{-1}\{y\}$ is closed for every $y\in Y$. It now follows from the observation

$$C(f:x)=\bigcap\{f(\mathrm{cl}(U)):U\in N(x)\}$$

that $C(f;(x))$ for every $x\in X$. The result now follows from Theorem 2.1 and the Closed Graph Theorem [9, page 50].

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