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HYPERSURFACES OF A SASAKIAN MANIFOLD WITH ANTINORMAL (f, g, u, v, λ)-STRUCTURE I

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Introduction

K. Yano, and U-Hang Ki [1] introduced the concept of antinormality of (f, g, u, v, λ)-structure and obtained many useful results. Lim, J.K. and Choe, Y.W. [2] also defined the antinormality of (f, g, u, v, λ)-structure and investigated the necessary and sufficient condition for (f, g, u, v, λ)-structure to be antinormal.

The purpose of the present paper is to obtain few more conditions for antinormality of (f, g, u, v, λ)-structure.

1. Preliminaries

We consider a C^∞ differentiable manifold with an (f, g, u, v, λ)-structure, that is, a Riemannian manifold with metric tensor g which admits a tensor field f of type (1,1), two 1-forms u and v and a function λ satisfying [3]

(1.1) $$\overline{\overline{X}}+X=u(X)U+vXV,\ f(X) \overset{\text{def}}{=\!=} \overline{X},$$

(1.2) $$g(\overline{X},\overline{Y})=g(X,Y)-u(X)u(Y)-v(X)v(Y),$$

(1.3) $$u(\overline{X})=\lambda v(X),$$

(1.4) $$v(\overline{X})=-\lambda u(X),$$

(1.5) $$\overline{U}=-\lambda V,$$

(1.6) $$\overline{V}=\lambda U,$$

(1.7) $$u(U)=1-\lambda^2=v(V),$$

and

(1.8) $$u(V)=0=v(U),$$

for arbitrary vector fields X and Y, U and V being vector fields defined by $u(X)=g(U,X)$ and $v(X)=g(V,X)$ respectively. If the tensor defined by

(1.9) $$S(X,Y)=N(X,Y)+du(X,Y)U+dv(X,Y)V,$$

$N(X,Y)$ being the Nijenhuis tensor formed with f, vanishes, the (f, g, u, v, λ)-structure is said to be normal [3].

We put

(1.10) $[f,f]\ (X,Y)=(D_{\overline{X}}u)(Y)-(D_{\overline{Y}}u)(X)-u\{(D_Xf)Y)-(D_Yf)(X)\}$
$+\lambda\{(D_Xv)(Y)-(D_Yv)(X)\},$

where D is the Riemannian connexion. If the tensor $[f,f]$ vanishes, the (f,g,u,v,λ)-structure is said to be antinormal [2].

2. Hypersurfaces of a Sasakian Manifold

Let M be an orientable hypersurface of a Sasakian manifold $\tilde{M}^{2n+1}$. Then there is an (f,g,u,v,λ)-structure induced in M, having the following properties [2].

(2.1) $(D_Yf)(Y)=-g(X,Y)U+u(Y)X-K(X,Y)V+k(X)v(Y),$

(2.2) $(D_Xu)(Y)=F(X,Y)-\lambda K(X,Y),$

(2.3) $(D_Xv)(Y)=-K(X,\overline{Y})+\lambda g(X,Y)$

and

(2.4) $D_X\lambda=K(X,U)-v(X),$

where $K(X,Y)=g(k(X),Y)$ is the second fundamental tensor in hypersurface M relative to $\tilde{M}^{2n+1}$.

Substituting (2.1), (2.2) and (2.3) into (1.10), we find,

(2.5) $[f,f]\ (X,Y)=K(Y,U)v(X)-K(X,U)v(Y).$

Using (2.4) in (2.5), we get [2]

(2.6) $[f,f]\ (X,Y)=(D_Y\lambda)v(X)-(D_X\lambda)v(Y).$

From (2.6) it is obvious that if λ is a constant, the induced (f,g,u,v,λ)-structure on M is antinormal. From [2], we know the following:

THEOREM A. *Let M be an orientable hypersurface of a Sasakian manifold such that the function λ is not a constant. In order that the induced (f,g,u,v,λ)-structure be antinormal it is necessary and sufficient that $kf+fk=0$.*

The condition $kf+fk=0$ is equivalent to

(2.7) $k(\overline{X})+\overline{k(X)}=0$

and

(2.8) $K(\overline{X},Y)-K(X,\overline{Y})=0.$

And from [1], we know the following:

THEOREM B. *In an orientable hypersurface M with an (f,g,u,v,λ)-structure of a Sasakian manifold such that $\lambda(1-\lambda^2)$ is almost every-where non-zero, the conditions*

(2.9) $K(\overline{X},Y)-K(X,\overline{Y})=0$

and

(2.10) $$S(X,Y)=2v(X)\{D_Y V-\lambda Y\}-2v(Y)\{D_X V-\lambda X\}$$

are equivalent.

Combining Theorem [A] and Theorem [B] and using (2.3), we have

THEOREM (2.1). *In an orientable hypersurface M of a Sasakian manifold, $\lambda \fallingdotseq$ constant, the induced (f, g, u, v, λ)-structure is antinormal if and only if*

(2.11) $S(X,Y)=2v(X)\ \overline{kY}-2v(Y)\ \overline{kX}$.

3. Some theorems on antinormal (f, g, u, v, λ)-structure

THEOREM 3.1. *In an orientable hypersurface M, $\lambda \neq 0$, if the Nijenhuis tensor vanishes the induced (f, g, u, v, λ)-structure is antinormal.*

PROOF. Let the Nijenhuis tensor of M vanishes, we have

(3.1) $$N(X,Y)=(D_{\overline{X}}f)(Y)-(D_{\overline{Y}}f)(X)-\overline{(D_X f)(Y)}+\overline{(D_Y f)(X)}=0.$$

Using (2.1) in above, we get

(3.2) $$\{K(\overline{Y},X)-K(\overline{X},\ Y)\}V+\{k(\overline{X})-\overline{k(\overline{X})}\}v(Y)-\{k(\overline{Y})-\overline{k(\overline{Y})}\}v(X)$$
$$-2g(\overline{X},Y)U=0.$$

Contracting (3.2) with respect to X and using (1.4), (1.5) and (1.6), we get

$$-2\lambda(K(U,X)-v(X))=0.$$

By virtue of (2.4) the above equation reduces to

(3.3) $$D_X\lambda=0,$$

that is λ is a constant and hence the structure is antinormal.

THEOREM 3.2. *In an orientable hypersurface M, $\lambda \neq$ constant, the induced (f, g, u, v, λ)-structure is antinormal if and only if*

(3.4) $$(D_{\overline{X}}F)(Y,Z)+(D_{\overline{Y}}F)(Z,X)+(D_{\overline{Z}}F)(X,Y)=2\{F(Y,X)u(Z)$$
$$+F(Z,Y)u(X)+F(X,Z)u(Y)\},$$

where $F(X,Y)\overset{\text{def}}{=\!=}g(\overline{X},Y)$.

PROOF. From (2.1), we have

(3.5) $$(D_{\overline{X}}F)(Y,Z)=F(Y,X)u(Z)+F(X,Z)u(Y)-K(\overline{X},Y)v(Z)+K(\overline{X},Z)v(Y).$$

Taking cyclic permutation of (3.5), we get

(3.6) $$(D_{\overline{X}}F)(Y,Z)+(D_{\overline{Y}}F)(Z,X)+(D_{\overline{Z}}F)(X,Y)=2\{F(Y,X)u(Z)+F(Z,Y)u(X)$$
$$+F(X,Z)u(Y)\}+v(X)\{K(\overline{Z},Y)-K(\overline{Y},Z)\}+v(Y)\{K(\overline{X},Z)$$
$$-K(\overline{Z},X)\}+v(Z)\{K(\overline{Y},X)-K(\overline{X},Y)\}.$$

Using Theorem [A] in (3.6), we get the result.

THEOREM 3.3. *In an orientable hypersurface M, $\lambda \neq$ constant, the induced*

(f,g,u,v,λ) is antinormal if and only if

$$(3.7)\quad 'N(X,Y,Z)+'N(Y,Z,X)+'N(Z,X,Y)=(D_{\overline{X}}F)(Y,Z)+(D_{\overline{Y}}F)(Z,X)+(D_{\overline{Z}}F)(X,Y),$$

where $'N(X,Y,Z)\overset{\text{def}}{=\!=}g(N(X,Y),Z)$.

PROOF. With the help of (2.1), we have

$$(3.8)\quad 'N(X,Y,Z)=\{K(\overline{Y},X)-K(\overline{X},Y)\}v(Z)+\{K(\overline{X},Z)+K(X,\overline{Z})\}v(Y)-\{K(\overline{Y},Z)+K(Y,\overline{Z})\}v(X)-2F(X,Y)u(Z).$$

Which gives

$$(3.9)\quad 'N(X,Y,Z)+'N(Y,Z,X)+'N(Z,X,Y)=\{K(\overline{Y},X)-K(\overline{X},Y)\}v(Z)+\{K(\overline{Z},Y)-K(\overline{Y},Z)\}v(X)+\{K(\overline{X},Z)-K(\overline{Z},X)\}v(Y)+2\{F(Y,X)u(Z)+F(Z,Y)u(X)+F(X,Z)u(Y)\}.$$

In view of Theorem [A] and Theorem (3.2), (3.9) proves the statement.

PROPOSITION 3.1. *In an orientable hypersurface M, the induced (f,g,u,v,λ)-structure is antinormal if and only if*

$$(3.10)\quad (D_XF)\ (Y,U)=(D_YF)\ (X,U).$$

PROOF. From (2.1) and (2.5), we get

$$(D_XF)(Y,U)-(D_YF)(X,U)=[f,f](Y,X).$$

which proves the result.

THEOREM 3.4. *In an orientable hypersurface M, $\lambda\neq$ constant, the induced (f,g,u,v,λ)-structure is antinormal if and only if*

$$(3.11)\quad 'S(X,Y,V)='N(X,Y,V)=(D_{\overline{X}}F)(Y,V)-(D_XF))(\overline{Y},V),$$

where $'S(X,Y,Z)\overset{\text{def}}{=\!=}g(S(X,Y),Z)$,

PROOF. Transvecting (1.9) with V and using (1.8), (2.3) and Theorem [A], we get

$$S(X,Y,V)='N(X,Y,V).$$

From (3.1), we have

$$(3.12)\quad 'N(X,Y,V)=(D_{\overline{X}}F)(Y,V)-(D_{\overline{Y}}F)(X,V)+(D_XF)(Y,\overline{V})-(D_YF)(X,\overline{V}).$$

which, due to (3.10), implies (3.11).

PROPOSITION 3.2. *In an orientable hypersurface M, $\lambda\neq0$ and $\lambda\neq$ constant, the induced $(f,g,u.v,\lambda)$-structure is antinormal if and only if*

$$(3.13)\qquad 'S(X,Y,V)=0='N(X,Y,V).$$

PROOF. Transvecting (2.11) with V and using (1.6), (2.5), we get

(3.14) $$'S(X, Y, V)=2\lambda[f, f](X, Y),$$
which proves the statement.

COROLLARY 3.1. *In an orientable hypersurface M, $\lambda\neq 0$ and $\lambda\neq$ constant, the induced (f, g, u, v, λ)-structure is antinormal if and only if*

(3.15) $$(D_{\overline{X}}F)(Y, V)=(D_XF)(\overline{Y}, V).$$

PROOF. Just follows from (3.11) and (3.13).

In an orientable hypersurface M, $\lambda\neq$ constant, with antinormal (f, g, u, v, λ)-structure, we know [1]

(3.16) $$K(X, V)=\beta u(X)$$
and

(3.17) $$K(X, U)=\beta v(X),$$
where

(3.18) $$\beta=\frac{K(U, V)}{1-\lambda_2}$$

THEOREM 3.5 *β, as given in* (3.18), *is constant if and only if*

(3.19) $K(X, k\overline{Y}=\beta F(X, Y)$.

PROOF. Differentiating (3.17) and using (2.2), (2.3), (2.8) and

(3.20) $$(D_XK)(Y, Z)-(D_YK)(X, Z)=0,$$
we get,
$$(D_X\beta)v(Y)=(D_Y\beta)v(X)$$
which, on putting $Y=V$, gives
$$(D_X\beta)(1-\lambda^2)=(D_V\beta)v(X),$$
that is $D_X\beta$ is proportional to $v(X)$ and hence we can write

(3.21) $$D_X\beta=\rho v(X),$$
where ρ is a function.

Now differentiating (3.16) and using (2.2), (2.3), (2.8), (3.20) and (3.21), we get

(3.22) $$\rho\{v(X)u(Y)-v(Y)u(X)\}=2\beta(F(X, Y)-2K(X, k\overline{Y})$$
From (3.22) it is clear that $\rho=0$, that is, $\beta=$constant if and only if (3.19) holds.

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