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TRANSFINITE SEPARATION AXIOMS IN TOPOLOGICAL SPACES

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1. Introduction

Using finite induction, Viglino [6] defined a new collection of separation axioms each of which implies Hausdorff separation and in turn is implied by complete Hausdorff (or functional Hausdorff) separation as follows: A topological space X is $\overline{T}_n (n \in N \equiv$natural numbers) if whenever x and y are distinct points of X, there is a collection of open sets $\{U_i : i=0, 1, \cdots, n-1\}$ such that

1. for $0 \leq i < n, x \in U_i$ and $y \notin \mathrm{cl}(U_i)$ [cl≡closure],
2. $\mathrm{cl}(U_{i-1}) \subset U_i$ for $1 \leq i < n$.

In the same paper, Viglino presented examples to show that there are spaces which are $\overline{T}_n$ but not $\overline{T}_{n+1}$ for every finite number n, and also that there are nonregular spaces which are $\overline{T}_n$ for every n. In Section 2 of this paper we generalize Viglino's definitions using ordertype indices, and then present examples to show that for ordinal number indices the classes of topological spaces so defined are distinct. Example 3 of this section not only generalizes the corresponding example due to Viglino, but is considerably simpler. In the remainder of the paper we investigate the relationships between these new separation axioms and the standard separation axioms of Hausdorff, Tychonoff, and Urysohn and construct topological extensions of spaces endowed with these new separation axioms which do not disturb the dispersion character of the spaces.

In this section we make those definitions needed in the remainder of the paper, explore the more elementary relationships concerning these definitions and present three examples illustrating the topological inequivalence of the new properties.

If α is an order type, a topological space X is defined to be $\overline{T}_\alpha$ if whenever x and y are distinct points of X, there is a collection of open sets $\{U_\beta : \beta < \alpha\}$ such that

1. for $0 \leq \beta < \alpha$, $x \in U_\beta$, and $y \notin \mathrm{cl}(U_\beta)$,
2. $\mathrm{cl}(U_\beta) \subset U_\gamma$ whenever $0 \leq \beta < \gamma < \alpha$.

Such a collection of open sets will be called a $\overline{T}_\alpha$*-chain* containing x and excluding y. A topological space X is defined to be B_α if whenever x and y are distinct points of X there is a neighborhood basis $\{U_\beta : \beta<\alpha\}$ at x for which $\mathrm{cl}\,(U_\beta)\subset U_\gamma$ whenever $0\leq\beta<\gamma<\alpha$. The space X is S_α if whenever $x\in X$ and U is an open set containing x, there is a collection of neighborhoods of x, $\{U_\beta : \beta<\alpha\}$ such that $\mathrm{cl}(U_\beta)\subset U_\gamma\subset U$ for $0\leq\beta<\gamma<\alpha$. We denote the order types of the sets of natural numbers, rational numbers, and real numbers with their usual ordering by ω, η, and λ respectively. If α is an ordertype $|\alpha|$ will denote the cardinality of α. Definitions of the separation axiom of Hausdorff (and of the remaining separation axioms refered to in this paper) can be found in [1]. A topological space X is *Urysohn* if whenever x and y are distinct points of X, there are corresponding open sets U and V such that $x\in U$, $y\in V$, and $\mathrm{cl}(U)\cap\mathrm{cl}(V)=\phi$. A topological space X is defined to be *completely Hausdorff* if whenever x and y are distinct points of X there is a continuous function f from X into the continuum $[0, 1]$ such that $f(x)=0$ and $f(y)=1$. Certain immediate consequences of our definitions are listed below.

1. The $\overline{T}_1$ and $\overline{T}_2$ separation axioms are equivalent to the separation axioms of Hausdorff and Urysohn.
2. If the T_1 space X is S_α then X is $\overline{T}_\alpha$. Also, if X is B_α then X is S_α.
3. If $|\alpha|\geq 1$ and the T_1 space X is S_α, then X is regular (i.e. T_3).
4. If the topological space X is $\overline{T}_\eta$ (respectively B_η or S_η) then X is $\overline{T}_\alpha$ (respectively B_α or S_α) for every countable ordertype α.

The first two examples we present indicate that one cannot infer a stronger separation property from a weaker separation property and that this degree of distinction is quite independent of the ordertype index. The third example provides at the same time a generalization of an example given by Viglino [6] and a much simpler construction which yields this generalized result.

EXAMPLE 1. *There is a space X which is S_η at every point but is not B_α for any countable ordertype α.*

PROOF. Let X be the Euclidean plane with one line identified to a point. Then X is S_η at every point but as X is not first countable [1] it is not B_α for any countable ordertype α.

EXAMPLE 2. *There is a topological space which is $\overline{T}_\eta$ but is not S_α for any nonzero ordertype α.*

PROOF. Let X be the set of real numbers and let a subbasis for the topology on X consist of {open intervals}$\cup Q$ [$Q\equiv$rationals]. Then X is $\overline{T}_\eta$ but X is not regular. The conclusion follows.

For example 3 we adapt a technique by P. Roy [4].

EXAMPLE 3. *Given any ordinal $\alpha>0$ there is a T_2 topological space which is $\overline{T}_\alpha$ but not $\overline{T}_{\alpha+1}$.*

PROOF. Let X be a T_2 topological space of little inductive dimension zero having the property that there are $2|\alpha|$ disjoint dense subsets of X. (If $|\alpha|\leq$ let X be the irrationals for example.) First, we order these disjoint dense subsets of X as $\{X_\beta:0\leq\beta<2\alpha\}$, and then define $X_{2\alpha}=X_0$. Let $Y=\bigcup_{\beta\leq 2\alpha}(X_\beta\times\{\beta\})$. We define a topology on Y by defining a neighborhood system at every point (y,β) of Y which uses both the original topology from X and the ordinal level β at which (y,β) resides. Let U be an open set in X which contains y.

i. If β is odd $(0<\beta<2\alpha)$, define $U^*=\{(x,\sigma):x\in U\cap X_\sigma$ and $\beta-1\leq\sigma\leq\beta+1\}$.

ii. If β is even and $0<\beta\leq 2\alpha$, define $U_\gamma{}^*=\{(x,\sigma):x\in U\cap X_\sigma$ and $\gamma<\sigma\leq\beta\}$ where γ is an ordinal less than β. Such sets will be refered to as sets of the form U^*.

iii. If $\beta=0$, define $U^*=\{(x,0):x\in U\cap X_0\}$.

Then a neighborhood basis of (y,β) in Y consists of sets of the form U^* where U is a neighborhood of y in X.

We must now demonstrate that Y has the desired properties. Let (y_1,β_1) and (y_2,β_2) be distinct points in Y. As X is both T_2 and of little inductive dimension zero, it follows readily that Y is $\overline{T}_\alpha$ between (y_1,β_1) and (y_2,β_2) if $y_1\neq y_2$. Hence we may assume $y_1=y_2$ and we denote their common value by y.

As $(y,\beta_1)\neq(y,\beta_2)$ it also follows that $\beta_1=0$ and $\beta_2=2\alpha$. If $U_\gamma=\{(x,\sigma):\ \sigma\leq 2\gamma\}$, then $\{U_\gamma:0\leq\gamma<\alpha\}$ is a $\overline{T}_\alpha$-chain about $(y,0)$ which excludes $(y,2\alpha)$. To show there is no $\overline{T}_{\alpha+1}$-chain about $(y,0)$ which excludes $(y,2\alpha)$ we assume there is such a chain $\{V_\gamma:0\leq\gamma\leq\alpha\}$ and obtain a contradiction. As V_0 is an open subset of Y containing $(y,0)$ there is an open set V' in X containing y such that $(V'\cap X_0)\times\{0\}\subset V_0$. But, as X_1 is dense in X, and $\mathrm{cl}(V_0)\subset V_1$, it follows that $(V'\cap X_1)\times\{1\}$

$\subset V_1$. Inductively, then if follows that $(V'\cap X_\gamma)\times\{\gamma\}\subset V_\gamma$ for $0\leq\gamma\leq 2\alpha$. However, $V'\cap X_{2\alpha}=V'\cap X_0$ and hence contains y. This shows that $(y, 2\alpha)\in V_{2\alpha}$ and this contradiction finishes the proof that Y is not $\overline{T}_{\alpha+1}$.

To close this section we prove three propositions which indicate some relationships between these new separation axioms and the standard sepraration axioms.

PROPOSITION 1. *A space X is completely Hausdorff if and only if X is $\overline{T}$.*

PROOF. Let x_1 and x_2 be distinct points of X. If X is completely Hausdorff, then there is a continuous function $f:X\to[0,1]$ such that $f(x_1)=0$ and $f(x_2)=1$. Let $U_r=f^{-1}([0,r))$ where $r\in Q\cap(0, 1)$. As f is continuous and $Q\cap(0, 1)$ has ordertype η, $\{U_r : r\in Q\cap(0,1)\}$ satisfies the properties required of a space to be $\overline{T}_\eta$ between x_1 and x_2. As x_1 and x_2 were arbitrary, the necessity is proved.

On the other hand, suppose that X is $\overline{T}_\eta$ and let x_1 and x_2 be distinct points of X. As X is $\overline{T}_\eta$ there is a collection of open sets $\{U_q : q\in Q\cap(0,1)\}$ such that

i. $x_1\in U_q$ for $q\in Q\cap(0,1)$,

ii. $x_2\notin \mathrm{cl}(U_q)$ for $q\in Q\cap(0,1)$, and

iii. if $q_1<q_2$ and both q_1 and q_2 are in $Q\cap(0,1)$, then $\mathrm{cl}(U_{q_1})\subseteq U_{q_2}$.

Adapting a technique used to prove Urysohn's chanacterization of normality we define $f: X\to[0,1]$ as

$$f(x)=\begin{cases}\inf\{q : q\in U_q\} & \text{if } x\in U_1,\\ 1 & \text{if } x\notin U_1.\end{cases}$$

Then f is continuous and $f(x_1)=0$ while $f(x_2)=1$. This completes the proof of Proposition 1.

Using Proposition 1 and the evident fact that if X is $\overline{T}_\lambda$ then X is $\overline{T}_\eta$ we have the following corollary.

COROLLARY 1.1. *A space is $\overline{T}_\eta$ if and only if it is $\overline{T}_\lambda$.*

PROPOSITION 2. *A space X is B_η if and only if X is first countable and completely regular.*

PROOF. Suppose X is B_η, let $x\in X$ and let K be a closed subset of X not containing x. As X is B_η, there is a base at x, $\{U_q : q\in Q\cap(0,1)\}$, such that

1. $U_q\subset X-K$ for every $q\in Q\cap(0,1)$, and
2. $\mathrm{cl}(U_{q_1})\subset U_{q_2}$ if $q_1, q_2\in Q\cap(0,1)$ and $q_1<q_2$.

If we define $f: X \to [0,1]$ exactly as in the proof of Proposition 1, then f is continuous and $f(x)=0$ while $f(K)=1$. Finally, the base at x, $\{U_q : q \in Q \cap (0,1)\}$, is countable and the necessity is proved.

Now, suppose X is completely regular and first countable, and let $x \in X$. Then there exists a countable base $\{U_n : n=1,2,\cdots\}$ at x for which $\mathrm{cl}(U_{n+1}) \subset U_n$ for $n=1,2,\cdots$ As X is completely regular, for each $n=1,2,\cdots$ there is a continuous function $f_n : X \to [0,1]$ such that $f_n(x)=0$ and $f_n(X-U_n)=1$. Define $f: X \to [0,1]$ by

$$f(y) = \sum_{n=1}^{\infty} f_n(y)/2^n.$$

Then f is continuous and $f(x)=0$. If we let $U_q = f^{-1}([0,q))$ for $q \in Q \cap (0,1)$ then $\{U_q : q \in Q \cap (0,1)\}$ is a base at x of order type η, and if $q_1 < q_2$ then $\mathrm{cl}(U_{q_1}) \subset U_{q_2}$. Consequently, X is B_η at x and the proof is complete.

The analogue of propositions 1 and 2 for the separation axiom S_η is Proposition 3 below and the proof is similar to those of the two aforestated propositions.

PROPOSITION 3. *A space X is S_η if and only if X is completely regular.*

Propositions 2 and 3 can be combined to yield the following result.

PROPOSITION 4. *A space X is B_η if and only if X is S_η and B_ω.*

3. Expansions of $\overline{T}_\alpha$ spaces

In [2] E. Hewitt defined the notion of the *dispersion character* of a topological space X to be the least cardinal number of a non-void open subset of X and denoted the dispersion character of X by $\Delta(X)$. In the same paper, properties of topological spaces were investigated relative to expansions of those spaces which retained the dispersion character of the original spaces. (If X_1 and X_2 are topological spaces with the same underlying set such that the topology from X_1 is contained in the topology from X_2, then X_2 is said to be an *expansion* of X_1.) In particular, Hewitt notes that many of the usual topological properties are not retained by expansions [Theorem 3] and suggests the question of whether a Urysohn space can be expanded to a regular space without changing the dispersion character. M. Powderly [3] answered this question in the negative. In this section we answer a natural question by showing that if X_1 is a T_2 topological space then there is an expansion X_2 of X_1 which is $\overline{T}_\alpha$ for every α, and $\Delta(X_1) = \Delta(X_2)$.

PROPOSITION 5. *Every T_2 topological space can be expanded to a topological space which is $\overline{T}_\alpha$ for every α without changing the dispersion character.*

PROOF. Consider the set E of all expansions X^* of X such that $\Delta(X^*)=\Delta(X)$. If X_1 and X_2 are in E, we define $X_1<X_2$ if X_2 is an expansion of X_1. Then $(E, <)$ is a partially ordered set and we let C be a chain in E. If $X^* \in C$ we denote the topology on X^* by $T(X^*)$. Then as C is a chain, $\bigcup_{X^* \in C} T(X^*)$ is a basis for a topology on the set X. Further, if $U \in \bigcup_{X^* \in C} T(X^*)$ then the cardinality of U is at least as large as $\Delta(X)$ because $\Delta(X^*)=\Delta(X)$ for every $X^* \in E$. Hence the topological space generated by $\bigcup_{X^* \in C} T(X^*)$ on the set X is a maximal element of C. It follows from the Hausdorff Maximal Principle that $(E, <)$ has maximal elements and we let $\hat{X}$ denote such a maximal element. We will show that if x_1 and $x_2 \in \hat{X}$, there is an open set U in $\hat{X}$ such that $x_1 \in U, x_2 \notin \mathrm{cl}(U)$ and $U=\mathrm{cl}(U)$. Suppose to the contrary that there are points x_1 and x_2 in $\hat{X}$ such that no such open set U exists. As X is T_2 there is an open set V in X such that $x_1 \in V$ and $x_2 \notin V$, and as $\hat{X}$ is an expansion of X, V is also open in $\hat{X}$. Define a new topology on the set X as

$$T^*=\{W_1 \cup (\mathrm{cl}(V) \cap W_2) : W_i \in T(\hat{X}) \text{ for } i=1,2\}.$$

The set T^* is a topology on the set X and the resulting topological space X^* is a proper expansion on $\hat{X}$. However, if W is open in $\hat{X}$ and $V \cap W$ is a nonempty open set of $\hat{X}$ then the cardinal number of $V \cap W$ is at least as large as $\Delta(\hat{X})=\Delta(X)$. It follows that $\Delta(X^*)=\Delta(X)$ and this contradicts the maximality of $\hat{X}$. Finally, if α is an ordinal number we must show $\hat{X}$ is T_α. If x_1 and x_2 are in $\hat{X}$, then there is an open-closed set U such that $x_1 \in U$ and $x_2 \notin U$. If we define $U_\beta=U$ for $0 \leq \beta < \alpha$ then the sequence $\{U_\beta : 0 \leq \beta < \alpha\}$ is a required sequence separating x_1 and x_2 and the proposition is proved.

Although the expansion $\hat{X}$ of X in the previous proposition contains a plethora of a open-closed sets it is not necessarily of little inductive dimension zero. If it were, then X would be regular, but M. Powderly [3] has provided an example of Urysohn space which can not be expanded to a regular space without changing the dispersion character.

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